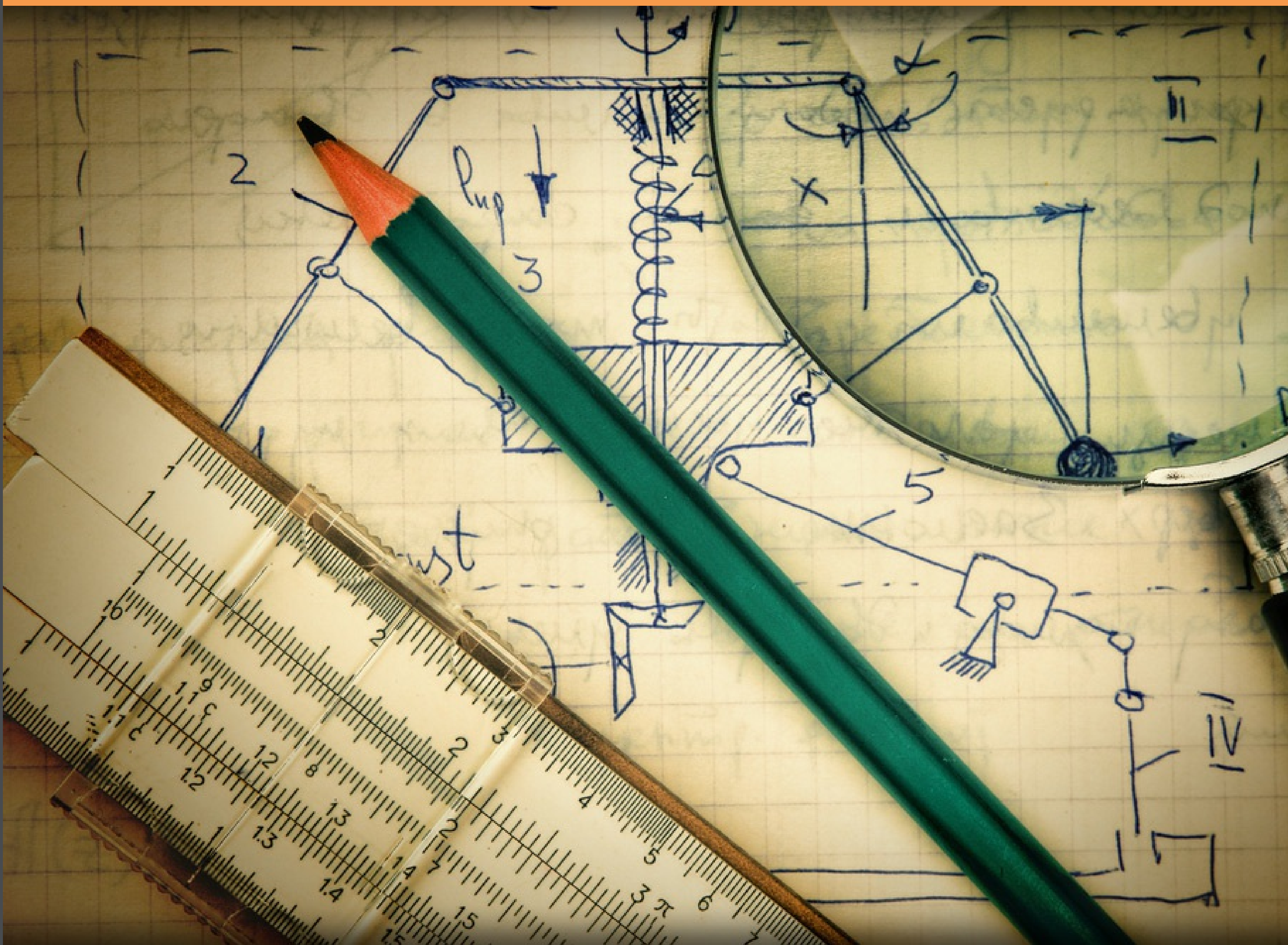
Foundation of Physics for Scientists and Engineers

Volume I: Mechanics, Heat and Sound

Ali R. Fazely



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Foundation of Physics for Scientists and Engineers

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Preface

A course in calculus-based Physics is a necessary part of the curriculum for engineers and scientists. The principal goal of such a course is to prepare students majoring in engineering and/or science for more advanced courses in these fields. A solid foundation in basic theories of physics is a must for completing a successful engineering or science curriculum. In this text, the emphasis will be on introducing the students to the fundamental concepts of physics and how different theories are developed from physical observations and phenomena. This textbook is written with minimal narratives and is geared more towards examples and problem solving techniques. The students will get a firsthand experience of how the theories in physics are applied to everyday problems in engineering and science. The learning outcome will be a broad knowledge and knowhow for problem solving techniques crucial in training engineers and scientists for a successful career in these fields.

Acknowledgments

All thanks to the Designer of laws.

Chapter 1

Physical Measurement and Units

Physics, by its very nature, is an empirical science. This means we need standard units by which we can measure different physical quantities. There are three basic physical quantities we are interested in length, mass and time. The reason behind this interest is that most other physical quantities in mechanics can be expressed in terms of these three basic quantities. For electricity we require current, for thermodynamics we need temperature, for light we need intensity, and finally we need amount of substance for chemical and physical processes.

1.1 The International System of Units (SI)

The International System of Units or SI for short, which is derived from the French term Le Système international d'unités is the modern metric system. As mentioned above, it is based on seven physical quantities; length, mass, time, current, temperature, intensity, and amount of substance. The system is fundamentally based on the three basic physical quantities of length, mass and time, i.e., *Meter* (m), *Kilogram* (kg), and *Second*(s). This system, known as the MKS was adopted and published in 1960. There is also another convention known as the CGS system, which is the acronym for *Centimeter* (cm), *Gram* (g), and *Second* (s).

The SI system is a decimal system, meaning it expands or contracts by factors of 10. This makes the SI system easy to manipulate and remember. The prefixes for the SI system are tabulated in table 1.1. The majority of these prefixes are derived from the Greek language.

The student should be mindful of the fact that he/she would probably not use most of these prefixes throughout his/her career. We have mentioned them here for the sake of completeness. The usual way of presenting or describing a very small or a very large number is by using exponents such as 10^{13} for a large number or 10^{-16} for a very small number. This way the reader would realize the magnitude without the need for memorizing and remembering a given prefix and its magnitude. The main goal of learning and mastering the SI system is to get a *feel* for the measurement of how large or small various objects or distances or weights or times are. The *feel* comes later on in the student's career when he/she learns this through experience.

Table 1.1: Prefixes, Language origin, Symbols and Magnitudes for the Short and the Long Scales in the *SI* system

Prefix	Origin	Symbol	Magnitude	Short/Long Scale	Year
yotta	Greek	Y	10^{24}	Sextillion/Trilliard	1991
zetta	Greek	Z	10^{21}	Sextillion/Trilliard	1991
exa	Greek	E	10^{18}	Quintillion/Trillion	1975
peta	Greek	P	10^{15}	Quadrillion/Billiard	1975
tera	Greek	T	10^{12}	Trillion/Billion	1960
giga	Greek	G	10^{9} Billion/Milliard	1960	
mega	Greek	M	10^{6} Million	1960	
kilo	Greek	K	10^{3}	Thousand	1795
hecto	Greek	H	10^{2}	Hundred	1795
deca	Greek	da	10	Ten	1795
			1	One	
deci	Latin	d	10^{-1}	Tenth	1795
centi	Latin	c	10^{-2}	Hundredth	1795
milli	Latin	m	10^{-3}	Thousandth	1795
micro	Greek	μ	10^{-6}	Millionth	1960
nano	Greek	n	10^{-9}	Billionth/Milliardth	1960
pico	Italian	p	10^{-12}	Trillionth/Billionth	1960
femto	Danish	f	10^{-15}	Quadrillionth/Billiardth	1964
atto	Danish	a	10^{-18}	Quintillionth/Trillionth	1964
zepto	French/Latin	z	10^{-21}	Sextillionth/Trilliardth	1991
yocto	Greek	y	10^{-24}	Sextillionth/Trilliardth	1991

The long and short scales are two of several different large-number naming systems used throughout the world for integer powers of ten. Many countries, including most in continental Europe, use the long scale whereas most English and Arabic speaking countries use the short scale. In each country, the number names are translated into the local language, but retain a name similarity due to shared origin. Some languages, particularly in East and South Asia, have large number naming systems that are different from the long and short scales. Long scale is the English translation of the French phrase échelle longue. It refers to a system of large-number names in which every new scale is a factor of one million (1000000) times larger than the previous term, i.e., a billion is a million millions or 10^{12}, and a trillion is a million billions or 10^{18}, etc. Short scale is the English translation of the French phrase échelle courte. It refers to a system of large-number names in which every new scale, greater than a million, is a factor of 1000 times greater than the previous scale; therefore, a billion is a thousand millions or 10^{9}, and a trillion is a thousand billions or 10^{12}, etc.

1.1.1 *SI* Unit of Length

The unit of length in the *SI* system is *meter*. Meter is defined as the distance light travels in vacuum in $\frac{1}{299792458}$ of a second. This definition makes the speed of light to be exactly 299792458 meters/second. Note metre is the internationally accepted spelling except for the USA where it is spelled *meter*. The following outline shows the historical evolution of the standard *SI* unit of length, the *meter*. On May 8, 1790, the French National Assembly defines the length of the new *meter* to be equal to the length of a pendulum with a period of

Figure 1.1: A computer generated picture of the original *meter* bar (source: Wikipedia).

two (2) seconds. Upon a recommendation by the French Academy of Science on March 30, 1791, the French National Assembly approves the new definition for the *meter* to be equal to one ten-millionth of the length of the Earth's meridian along a quadrant through Paris, which is the distance from the equator to the North Pole. On September 28, 1889, the first General Conference on Weights and Measures (CGPM) defined the *meter* as the distance between two lines on a standard bar of an alloy of platinum (Pt) with ten percent iridium (Ir), measured at the melting point of ice. Figure 1.1 is a computer generated picture of the original *meter* bar used as the standard *meter* stick from 1889 to 1960. Twenty-nine (29) copies were made and were sent to different countries to serve as their standard to measure length.

On October 6, 1927, the seventh CGPM adjusted the definition of the *meter* to be the distance, at $0\ {}^{\circ}C$ ($32\ {}^{\circ}F$), between the axes of the two central lines marked on the prototype bar of platinum-iridium, this bar being subject to one standard atmosphere of pressure and supported on two cylinders of at least $1\ cm$ ($0.39\ in$) diameter, symmetrically placed in the same horizontal plane at a distance of 571 millimeters ($22.5\ in$) from each other. On October 24, 1960, the 11^{th} CGPM defined the *meter* to be equal to 1650763.73 wavelengths in vacuum of the orange-red light emitted by the $krypton - 86$ atom when it De-excites from $^2p_{10}$ and $^5d_5$ quantum levels. On October 21, 1983, the 17^{th} CGPM defined the *meter* as equal to the distance traveled by light in vacuum in a time interval of $\frac{1}{299792458}$ of a second. In 2002, the International Committee for Weights and Measures (CIPM) recognized the *meter* to be a unit of proper length and recommended the 1983 definition to be restricted to “lengths l which are sufficiently short for the effects predicted by general relativity to be negligible with respect to the uncertainties of realization”.

The *meter* is specified by the letter m. The prefixes mentioned above with the latest standard definition of the *meter* provides general units we use in science, engineering and technology to appropriately express various distances, areas and volumes depending on how large or small they may be. For example, distances between two points on Earth or to the Sun and the Moon are usually expressed in terms of *kilometers* (km). Note another unit of length, used for very small scales (atomic scales), is the Angstrom (Å). It is one ten-billionth of a *meter* or $10^{-10}\ m$. For large distances, interstellar or intergalactic, distances are usually expressed in terms of *light year* ($l.y.$). One $l.y.$ corresponds to the distance light travels in one year. Its value in km is;

$1\ l.y. = 9.4605284 \times 10^{12}\ km.$

Another unit of length used is astronomy is *parsec* shown by the symbol pc. The definition of a pc is the distance at which Earth-Sun separation subtends an angle of one arc second. A pc is therefore $3.26163344\ l.y.$. For cosmological distances, kpc and Mpc are usually used.

Example 1. Calculate the area of a fenced yard $100\ m$ long and $50\ m$ wide. Express your answer in terms of square km.

Answer:

The area of a rectangle is the length times the width.

$A = l \times w,$
$A = 100\ m \times 50\ m = 5000\ m^2.$

Note, area is expressed in terms of square meters or m^2.

Now we want to convert $5000\ m^2$ to square kilometers or km^2. There are two ways we can do this:

First we can convert the length and the width to km and then multiply the results.

$l = 100\ m/1000\ m/km = 0.1\ km$
$w = 50\ m/1000\ m/km = 0.05\ km$
$A = 0.1 \times 0.05 = 5 \times 10^{-3}\ km^2$

For the second method, we find the number of square meters per square km. One km is 1000 m, therefore 1 $km^2 = 1000\ m \times 1000\ m = 1000000\ m^2$. Then the area is;

$A = 5000\ m^2 = 10000000\ m^2/km^2 = 5 \times 10^{-3}\ km^2$

Example 2. Calculate the volume of a raindrop with a radius of 2 mm. Express your answer in terms of mm^3. How many raindrops would fill a tank with a volume of 1 m^3?

Answer:

The volume of the sphere is;

$V = \frac{4}{3}\pi r^3$
$V = \frac{4}{3}3.14159 \times 2^3$
$V \approx 34\ mm^3$

A tank with a volume of 1 m^3 is $10^9\ mm^3$; therefore, the number of raindrops required to fill it up is;

Number of raindrops $= 10^9/34 \approx 29411764.7$

1.2 *SI Unit of Mass*

The *SI* unit of mass is kilogram (kg). The definition is the equivalent of 1 cubic decimeter of pure water at standard pressure and melting ice temperature. The standard kg shown in the computer generated figure 1.2 is made of an alloy 90% Pt and 10% Ir by weight.

The cylinder has equal height and diameter of 39.17 mm. Note the edges have four-angle chamfer to minimize wear. The International Prototype Kilogram (IPK) is kept at the Bureau International des Poids et Mesures (International Bureau of Weights and Measures) in *Sèrves* on the outskirts of Paris. The standard kg, accompanied by six other identical copies, are kept in an environmentally controlled vault. Forty duplicates were made and were sent to various countries for their weight standard. The alloy has ideal physical properties. Extreme high density of the alloy 21.186 g/cm^3, which is almost twice that of Lead, occupies a very small volume. Note the inch ruler for comparison. It also has satisfactory electrical and thermal conductivities, and low magnetic susceptibility. The various duplicates are compared to the standard kg every 50 years.

Figure 1.3 shows the mass drift over time of national prototypes K21 and K40, plus two of the IPK's sister copies, K32 and K8. All mass changes are relative to the IPK. The initial 1889 starting-value offsets relative to the IPK have been nulled. Shown in figure 1.3 are relative measurements only; no historical mass-measurement data is available to determine which of the

Figure 1.2: A computer generated picture of the original standard kilogram (source: Wikipedia).

prototypes has been most stable relative to an invariant mass in nature. Most likely all the prototypes have gained mass over a century and that K21, K35, K40, and the IPK have simply gained less than the others.

Example 3. The copies of the kg show variation in weight gain as much as 70 μ grams over a period of approximately 100 years. If the mass of the gunk build up on the kilogram replica as well as the original is due to hydrocarbon buildup and we assume it to be CH with a mass of 2.2×10^{-26} kg, calculate how many CH molecules have made their home on the IPK?

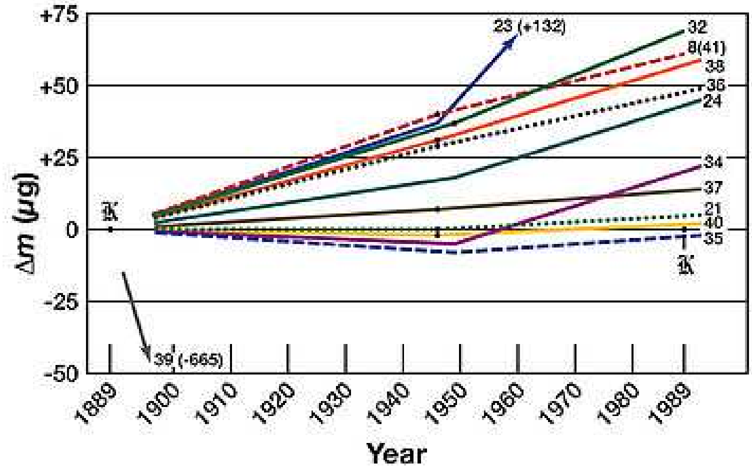

Figure 1.3: The plot of relative mass drift over a period of almost a century of all copies of the standard kilograms (source: Wikipedia).

Answer:

The mass gain of 70 μ g is 70×10^{-9} kg or 7×10^{-8} kg. Therefore, the number of CH molecules N, is;

$N = \frac{7\times10^{-8}kg}{2.2\times10^{-26}kg} \approx 3.2 \times 10^{18}$

1.3 Atomic Mass Unit

In the case of atomic, nuclear or particle physics, instead of expressing masses in the SI system, the Atomic Mass Unit or amu or u for short is often used. The measurement is based on the atomic mass of the ^{12}C defined, by international agreement, to be exactly 12 u. Then all other atoms or particles can be expressed in terms of this unit. For example, the mass of the proton is 1.007276 amu, the mass of the neutron is 1.008665 amu, and the mass of the electron is 0.0005485799 amu. Note; the mass of 1 u is $1.66053886 \pm 0.00000010 \times 10^{-27}$ kg.

Example 4. When neutrons and protons bind to form nuclei, they release energy. The amount of energy released is equal to the energy that binds the nucleus. A deuteron is a heavy hydrogen wherein a neutron and a proton are bound together. If the binding energy (BE) of the deuteron is 0.002362 u, find the mass of the deuteron.

Answer:

The mass of the deuteron is;
$M_d = M_p + M_n - BE$
$M_d = 1.007276\ u + 1.008665\ u - 0.002362\ u$
$\boxed{M_d = 2.015379\ u}$

1.4 SI Unit of Time

The unit of time in the SI system is *second*, usually denoted by s or sec. The definition of the second was, for almost 1000 years (1000 AD to 1960 AD), 1/86400 of a solar day. In the period of 1960-67 the second was defined as a fraction of the period of the Earth's orbit around the Sun in the year 1900. After 1967, the second has been defined as; the time corresponding to 9,192,631,770 periods of transition between hyperfine levels of the ^{133}Cs atom. The SI prefixes of table 1.1 are applied only for fractions of the second, such as millisecond ($1\ ms = 1/1000\ s$), micro-second ($1\ \mu s = 10^{-6}\ s$) and so on. For larger units, the traditional units such as minutes ($60\ s$), hour ($3600\ s$), day ($24\ h$), and so on are used.

Example 5. The lower limit of the proton lifetime is measured to be $1.01 \times 10^{34}\ years$. Calculate this number in ns. If the lifetime of a subatomic particle called the positive pion is 26 ns, what is the ratio of the lifetime of the proton to that of the pion?

Answer:

1 year is 365.24 days, then;
$1\ year = 365.24 \times 24 = 8764.8\ h$,
1 hour is 3600 seconds and
$1\ year = 8764.8\ h \times 3600\ s/h = 31553280\ s$.
$1\ year = 3.155328 \times 10^{16}\ ns$.

The lower limit of the proton lifetime is;

$\tau_p > 3.155328 \times 1016 \times 1.01 \times 10^{34}\ ns$.

Or;

$\boxed{\tau_p > 3.18688 \times 10^{50}\ ns}$.

Now we divide this lower limit of the proton lifetime by the mean lifetime of the positive pion.

$$\boxed{\frac{\tau_p}{\tau_{\pi+}} > 1.22572 \times 10^{49}}$$

In summary, unit of length in the SI system is *meter* (m) and is defined as the distance light travels in vacuum in 1/299792458 of a *second*. This definition forces the speed of light to be exactly $299792458\, m/s$.

Unit of mass in the SI system is *kilogram* (kg) and is the mass of the standard kilogram made from 90% Pt - 10% Ir alloy kept under strict environmental control in a museum in Paris, France. Atomic Mass Unit is defined as the mass of ^{12}C to be exactly $12\, u$ or amu. All other subatomic particles or nuclei are then compared to that of ^{12}C.

Unit of time in the SI system is *second* (s) and is defined as the time corresponding to the equivalent time of 9192631770 periods of transition between hyperfine levels of the ^{133}Cs atom.

1.5 Problems

1. The size of an Amoeba is 150 μm. Express this length in nm and $\AA$.

2. Find the surface area of Earth in km^2 if the diameter of the Earth is 12740 km. If the surface area of the Moon is 3.7915×10^7 km^2, find the Moon's radius and compare it with the radius of the Earth.

3. The size of the H-atom is 1 $\AA$. Assume the H-atom is spherical; calculate its volume in m^3.

4. The distance to the Earth's closest star, the Alpha Centauri binary star system, is 1.339 pc. Calculate this distance in $l.y.$ and in km.

5. The radius of a proton is 1 fm. Calculate its volume in fm^3 and in $\AA^3$. Compare the volume of the proton to that of the H-atom in problem 3.

6. A farm is estimated to be worth \$500000 US. If the farm is square in shape, and each side is 1000 m, calculate the cost of 1 m^2 of the farm.

7. Calculate the distance in m to our sister galaxy, the Andromeda galaxy, if when we look at it today it is a 2.5 million year old sight.

8. The eccentricity of the Earth's orbit around the Sun is 0.0167. If the relation between the semi-major axis and semi-minor axis is $b = a\sqrt{1-e^2}$ and $\frac{a+b}{2} = 150000000$ km, find the area of the orbital ellipse in m^2.

9. The radius of a ping pong ball is 2.0 cm. If non-uniformities as much as 100 μm exist on the surface of the ball and we blow it up to the size of the Earth, calculate the height of the highest peak in km and compare it with Mount Everest (8.848 km).

10. An ingot of gold the size of a lead brick (10.0 $cm \times 20.0$ $cm \times 5.0$ cm) has a mass of 19.32 kg. Find the density of gold in g/cm^3.

11. In problem 10, if we replace gold with lead and the density of lead is 0.01134 kg/cm^3; find the mass of the lead brick in grams.

12. If the mass of the Earth is 5.96×10^{24} kg and its radius is 6370 km, what is its density in g/cm^3?

13. If the mass of the Sun is 1.99×10^{30} kg , and a neutron star has a radius of 10 km and it is twice as massive as the Sun, what is its density in g/cm^3?

14. If the surface area of the Moon is 3.7915×10^7 km^2 and its mass is 7.348×10^{22} kg, calculate its density and compare it with that of the Earth found in problem 12.

15. The mass of the proton is 1.007276 u. Convert this number into grams and Picograms.

16. The radius of a proton is about 1 fm, using the results of problem 15; calculate the density of the proton in g/cm^3.

17. The binding energy of the H-atom is 1.8×10^{-8} u. The mass of the proton is 1.007276 u and that of the electron is 0.0005485799 u. Find the mass of the H-atom in kg.

18. Find the density of the H-atom from the results of problem 17 and assume its diameter is 1 $\AA$. Compare this density with that of the proton.

19. The lifetime of the positive pion, a subatomic particle, is 26 ns. Convert this lifetime to minutes.

20. Half-life of a given radioactive substance is defined as the time it takes for the substance to decay to half of its original mass. If the half-life of ^{60}Co is 5.7 years and we start with 1.0 g of ^{60}Co, how much ^{60}Co would we have left after three half-lives?

21. The mean-life of a free neutron is 15 m; convert this number to fs, ns and days.

22. The energy output of the Sun requires 2.8×10^{39} protons to undergo p-p fusion/second. Calculate how many protons have to fuse in the next 5 billion years.

23. In the scripture, Noah lived to be 950 years old. Compare Noah's lifespan to a subatomic particle called the muon having a mean life of 2.2 μs.

24. In Carbon dating of fossils, the amount of ^{14}C, a radioactive isotope of C is measured. By measuring the amount of ^{14}C in the fossil, researchers can estimate the age of the fossil. If in a given fossil, the amount of ^{14}C is found to be a quarter of that in similar living organisms, what is the age of the fossil? The half-life of ^{14}C is 5730 years.

Chapter 2

Scalars and Vectors

2.1 Physics a Mathematical Science

In Chapter one, we introduced Physics as an empirical science, however, once we perform our basic measurements, we then have to describe physical quantities by the means of mathematics. This requirement, therefore, makes Physics a Mathematical Science. Physical quantities can be classified in general into two categories; scalars and vectors. We will now define these two quantities which will provide the basis through which we can complete this course.

Definition of Scalar:*A physical quantity is a scalar when it is completely defined by a single number corresponding to its magnitude.*

Examples of scalars are mass, time and energy. The mathematical operations, such as addition, subtraction, multiplication and division are simple arithmetic operations that we all know from elementary school and are all applicable to scalar quantities.

Definition of Vector:*A physical quantity which is defined by two numbers corresponding to its magnitude and direction is a vector.*

Examples of vectors are displacement, velocity, acceleration and force to name a few. Vector manipulation and operations are more detailed than scalars because we have to consider both magnitude and direction. Addition and subtraction of vectors can be done in two ways: 1) the parallelogram method and 2) the component method.

2.2 Vector Addition and Subtraction

2.2.1 The Parallelogram Method

The parallelogram rule provides a method by which two or more vectors can be added or subtracted. In figure 2.1 two vectors **A** and **B** make an angle of θ, the resultant vector **R** is the diagonal where **R** starts at the intersect point O of the two vectors **A** and **B**. Note, we have drawn a line from the end of vector **B** parallel to the vector **A** and another line from the end of **A** parallel to the vector **B**, hence, creating a parallelogram. By the definition of parallelogram these two lines are equal to the two vectors **A** and **B** in magnitude and direction. Note the magnitude of the vector is graphically represented by its length.

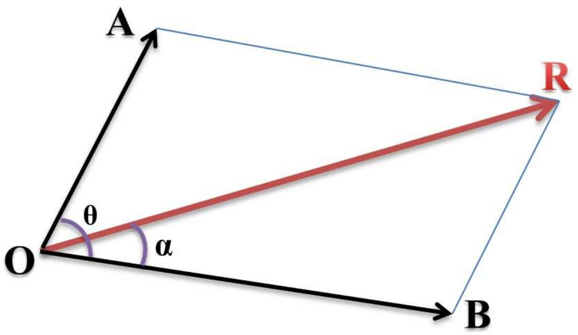

Figure 2.1: An example of the parallelogram rule for addition of two vectors.

The vector **R** is called the resultant vector and it has the same effect on the point O as the two vectors **A** and **B** combined. If there are additional vectors, we must then find the resultant vector **R** and a third vector **C** and so on.

The mathematical operation providing the magnitude of the resultant vector **R** comes from geometry and trigonometry. In the lower triangle formed by the vectors **B** and **R**, we can write the following identity known as *the Law of Cosines*.

$$R^2 = A^2 + B^2 + 2AB\cos\theta \tag{2.1}$$

Therefore, the magnitude of the resultant vector **R** is;

$$\boxed{R = \sqrt{(A^2 + B^2 + 2AB\cos\theta}}. \tag{2.2}$$

Now we must address the question regarding the direction of the vector **R**. The direction can be expressed in terms of the angle that the vector **R** makes with vector **B** which we call α. Then we can write;

$$\boxed{\alpha = \sin^{-1}(A\sin\theta)/R.} \tag{2.3}$$

We therefore have defined the resultant vector **R** with both its magnitude as well as its direction. The parallelogram rule can be extended to the polygon method for several vectors. The polygon method requires stringing all vectors head-to-tail and the resultant is the vector connecting the tail of the first vector in the string to the head of the last vector in the string. Figure 2.2 illustrates the polygon method.

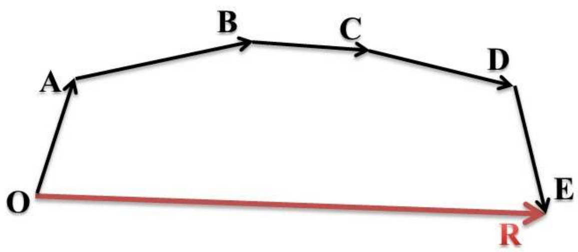

Figure 2.2: Shown here schematically is the polygon method for vector addition. The resultant vector **R** is the sum of all vectors **A**, B, C, D and **E**.

Subtraction of two vectors is performed when we are interested in a vector which can replace the difference of two vectors. We can again employ the parallelogram rule and in this case as shown in figure 2.3, the difference of the two vectors **A** and **B** is the other diagonal of the parallelogram.

The magnitude of the vector **R** is;

$$R^2 = A^2 + B^2 - 2AB\cos\theta. \tag{2.4}$$

$$\boxed{R = \sqrt{A^2 + B^2 - 2AB\cos\theta}}. \tag{2.5}$$

The angle α that vector **B** makes with the vector **R** is;

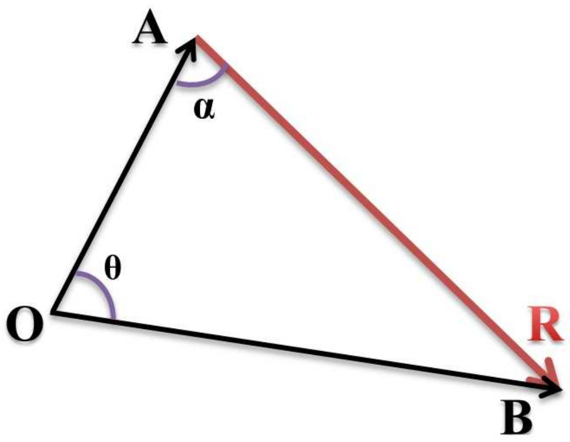

Figure 2.3: The difference vector **R** is simply the other diagonal of the parallelogram. Compare this figure to figure 2.1.

$$\boxed{\alpha = \sin^{-1}(B\sin\theta)/R.} \tag{2.6}$$

The methods we just described are not very practical in science and engineering. Scientists and engineers prefer to describe vectors within a fixed frame of reference so that they can communicate their results using a global coordinate system. We will now describe vector addition and subtraction using the component method.

2.3 The Component Method

The component method is a very attractive method for adding or subtracting vectors regardless of the number of vectors involved. We can also determine their resultant magnitude and direction with respect to a global xyz coordinate system. The component method as the name implies involves finding the component of a given vector with respect to a given coordinate system and thereby forcing the components to be along the same direction. The best way to describe this method is by the use of a graphic illustration. Figure 2.4 shows two vectors A and B.

The method, as shown in figure 2.4, breaks down each vector into its x and y components and we then add these components algebraically to obtain the x and y components of the resultant vector **R**, namely R_x and R_y. We now illustrate the algebra for obtaining these components.

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta. \tag{2.7}$$

And,

$$B_x = B\cos\theta \; and \; B_y = B\sin\theta. \tag{2.8}$$

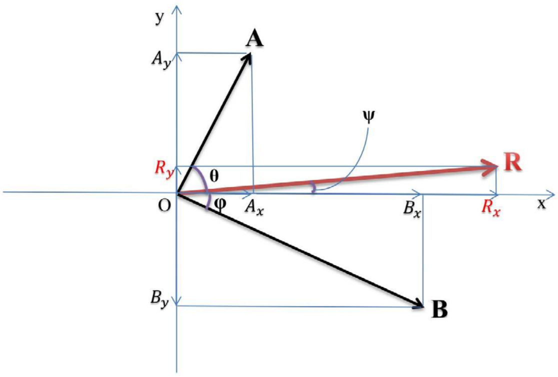

Figure 2.4: An illustration of the component method for vector addition.

The magnitudes and the directions of these components are now obvious and

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta. \tag{2.9}$$

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta. \tag{2.10}$$

can be expressed as simply their algebraic sums.

$$\begin{aligned} R_x &= A_x + B_x, \\ R_y &= A_y + B_y \end{aligned}$$

Therefore, from the Pythagorean Theorem we can write;

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta. \tag{2.11}$$

$$A_x = A\cos\theta \; and \; A_y = A\sin\theta. \tag{2.12}$$

$$|R| = \sqrt{R_x^2 + R_y^2}. \tag{2.13}$$

The two vertical lines bracketing **R** signify the magnitude of the vector. The direction of the vector **R** is then defined by the following equation;

$$\alpha = \tan^{-1}\frac{R_y}{R_x}. \tag{2.14}$$

Note, the angle α is with respect to the $x - axis$ rather than a given vector.

Example 1. A car travels exactly south-east 5 km and then exactly north-east for 7 km. Find the resultant displacement vector traveled by the car.

Answer:

Note, in the statement of the problem, we are just asked to find the resultant displacement of the car. We should understand this means both magnitude and the direction of the displacement vector.

South-east direction means a $-45°$ angle with the x-axis or east. North-east means a $45°$ angle with the x-axis. Therefore, we can write;

$$\begin{aligned} A_x &= 5\cos(-45°) = 5\sqrt{2}/2 \\ A_y &= 5\sin(-45°) = -5\sqrt{2}/2, \end{aligned}$$

and,

$$\begin{aligned} B_x &= 7\cos(45°) = 7\sqrt{2}/2 \\ B_y &= 7\sin(45°) = 7\sqrt{2}/2. \end{aligned}$$

The resultant vector can then be written as;

$$\begin{aligned} R_x &= 5\sqrt{2}/2 + 7\sqrt{2}/2 = 12\sqrt{2}/2 \\ R_y &= -5\sqrt{2}/2 + 7\sqrt{2}/2 = 2\sqrt{2}/2 \\ |R| &= \sqrt{R_x^2 + R_y^2} \\ &\boxed{|R| = \sqrt{72 + 2} \approx 8.6 \; km}. \end{aligned}$$

The angle α is;

$$\alpha = \tan^{-1}\frac{2\sqrt{2}/2}{12\sqrt{2}/2} = \tan^{-1}(1/6) \approx 9.46°. \tag{2.15}$$

2.4 Unit Vectors and Representations

A very popular way to represent a vector in various areas of physics and engineering is to represent it in terms of its unit vector components. The unit vectors **i**, **j** and **k** are unit vectors in the direction of the x, y and z axes respectively. A graphical depiction is shown in figure 2.5. In figure 2.5, the vector **A** can be represented by the magnitude of each component times the unit vector corresponding to that specific component. We can, therefore, express the vector **A** as,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}, \tag{2.16}$$

or;

$$\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}. \tag{2.17}$$

Addition of vectors with the **ijk** notation is straightforward. All that is required is to add each component to its counterpart algebraically.

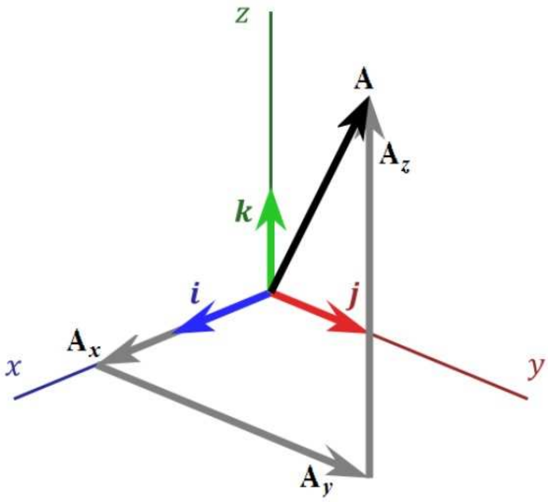

Figure 2.5: An illustration of expressing a vector using **ijk** representation.

Therefore,

$$\mathbf{R} = \mathbf{A} + \mathbf{B}, \tag{2.18}$$

or;

$$\mathbf{R} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} + B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$
$$\boxed{\mathbf{R} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_y)\mathbf{k}}.$$

The magnitude of the resultant vector **R** is;

$$\boxed{|\mathbf{R}| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}}. \tag{2.19}$$

For the direction a minimum of two angles are required;

$$\boxed{\theta = \tan^{-1}\frac{(A_y + B_y)}{(A_x + B_x)}}$$
$$\boxed{\psi = \tan^{-1}\frac{(A_z + B_z)}{(A_x + B_x)}}$$

Example 2. Find the resultant vector for the two vectors $\mathbf{A} = 5\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$.

Answer:

We simply add each component.

$\mathbf{R} = 5\mathbf{i} + 6\mathbf{j} - 4\mathbf{k} - 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
$\mathbf{R} = 3\mathbf{i} + 9\mathbf{j} - 7\mathbf{k}$.

The magnitude is;

$|\mathbf{R}| = \sqrt{(3^2 + 9^2 + .(-7).^2)} = \sqrt{139} \approx 11.79$
$\boxed{|\mathbf{R}| \approx 11.79}$,

and the direction;

$\theta = \tan^{-1} \frac{9}{3} = \tan^{-1} 3$
$\boxed{\theta \approx 71.57^\circ}$
$\psi = \tan^{-1} \frac{-7}{3}$
$\boxed{\psi \approx 113.20^\circ}$

2.5 Vector Multiplications

Vectors can be multiplied in two ways. Depending on the physical quantity one may require the *dot* or *scalar* product of two vectors and the *cross* or the *vector* product of two vectors.

2.5.1 Dot product of two vectors

Two vectors **F** and **d** making an angle θ can be multiplied to produce a scalar through a vector multiplication operation called the *dot* product, *scalar* product or *inner* product. Figure 2.6 shows the graphical representation of the above statement.

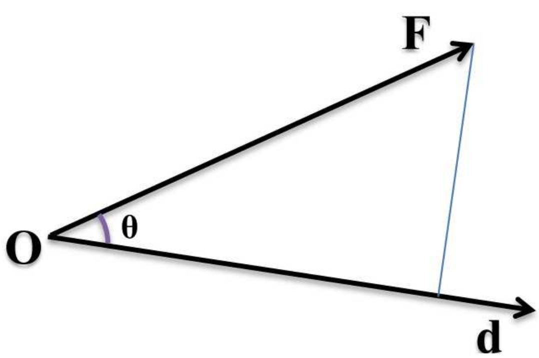

Figure 2.6: The scalar product of two vectors **F** and **d**.

The scalar product W of the two vectors **F** and **d** is therefore;

$$\boxed{W = \mathbf{F} \cdot \mathbf{d}}, \tag{2.20}$$

or;

$$\boxed{W = Fd\cos\theta} \tag{2.21}$$

The choice of W for the scalar product of the vectors **F** and **d** is deliberate. As we will see later on in this book, the physical quantity called work (W) is the dot product of the two vectors the force (**F**) and the displacement (**d**).

Example 3. Vectors **A** and **B** make an angle of 60°. Find the dot product of these two vectors.

Answer:

$C = \mathbf{A} \cdot \mathbf{B}$
$C = AB\cos\theta$

or;

$C = AB\cos 60°.$

$\boxed{C = AB(0.5) = 0.5AB}$

Scalar product of vectors can also be performed using the **ijk** vector representation. Let us define two vectors **A** and **B** in terms of its **ijk** components, i,e., $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$

The dot or scalar product of **A** and **B** is;

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \tag{2.22}$$

Note, in the above multiplication operation, $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = 0$, therefore;

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z} \tag{2.23}$$

Example 4. Calculate the dot product of the two vectors **A** and **B** of example 2.

Answer:

The two vectors **A** and **B** of example 2 are;

$\mathbf{A} = 5\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and
$\mathbf{B} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$.

$C = \mathbf{A} \cdot \mathbf{B} = (5)(-2) + (6)(3) + (-4)(-3),$

or;

$$\boxed{C = \mathbf{A} \cdot \mathbf{B} = 20}$$

2.5.2 Cross product of two vectors

The cross product or vector product of two vectors **A** and **B** yields another vector. Since the product is a vector, we must not only calculate its magnitude but also find its direction. The magnitude of the cross product vector **R** is;

$$\mathbf{R} = \mathbf{A} \times \mathbf{B}, \tag{2.24}$$

or;

$$\boxed{|\mathbf{R}| = \mathbf{AB} \sin \theta} \tag{2.25}$$

The direction of the vector **R** is along the line perpendicular to the plane created by the two vectors **A** and **B**. The sign of the vector will change if the order of the cross multiplication changes, i.e.

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{2.26}$$

The direction and the sign can be found easily by the so-called *right-hand-rule*. As shown in figure 2.7, point your index finger in the direction of the first vector, in this case vector **A**, and your middle finger in the direction of the second vector **B**, then your thumb will point to the direction of the vector product **R**. Note your index finger and middle finger make an angle θ together. If you point your index finger in the direction of the vector **B** and take it to be the first vector, then you have to turn your hand 180° in order to be able to point in the direction of the vector **A** with you middle finger, therefore, your thumb will point downward and that would be the direction of the vector **R**.

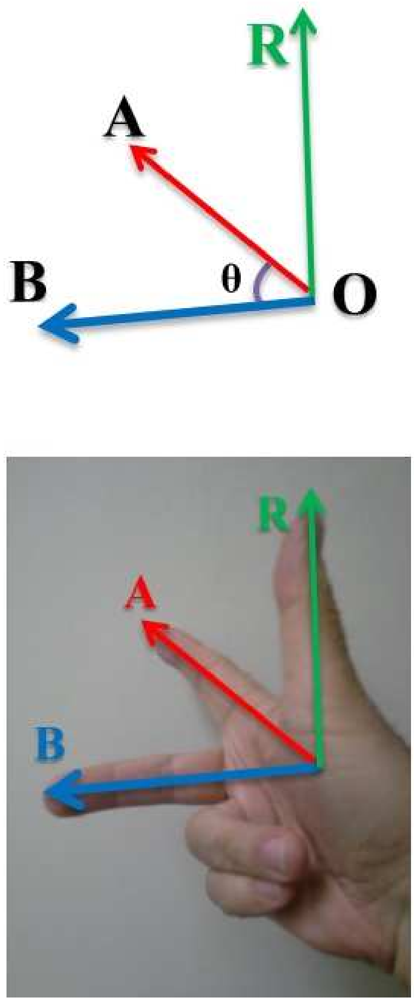

Figure 2.7: The right-hand rule for determining the direction of the cross product of two vectors.

Cross product of vectors can also be performed using the **ijk** vector representation. Let us go back to our two vectors, **A** and **B**, and their **ijk** representations, i.e., $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$.

According to equation 2.26, the magnitude of the cross product of two vectors is a function of the sine of the angle between them, namely $\sin\theta$. We know that if $\theta = 0$, then $\sin\theta = 0$ and if $\theta = 90$, then $\sin\theta = 1$. This implies that only cross products such as $\mathbf{i}\times\mathbf{j}=\mathbf{j}\times\mathbf{k}=\mathbf{i}\times\mathbf{k}$ will survive and $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}$ will vanish. We therefore can write;

$$\mathbf{A}\times\mathbf{B}=(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}) \tag{2.27}$$

or;

$$\boxed{\mathbf{R}=\mathbf{A}\times\mathbf{B}=(A_yB_z-A_zB_y)\mathbf{i}-(A_xB_z-A_zB_x)\mathbf{j}+(A_xB_y-A_yB_x)\mathbf{k}} \tag{2.28}$$

We can also write this cross product as the determinant of the following matrix.

$$\mathbf{A}\times\mathbf{B}=det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z.\end{pmatrix} \tag{2.29}$$

Or;

$$\mathbf{A}\times\mathbf{B}=\mathbf{i}det\begin{pmatrix}\mathbf{j} & \mathbf{k}\\ A_y & A_z\\ B_y & B_z\end{pmatrix}-\mathbf{j}det\begin{pmatrix}\mathbf{i} & \mathbf{k}\\ A_x & A_z\\ B_x & B_z\end{pmatrix}+\mathbf{k}det\begin{pmatrix}\mathbf{i} & \mathbf{k}\\ A_x & A_y\\ B_x & B_y\end{pmatrix} \tag{2.30}$$

Equation 2.30 yields equation 2.28 or the cross product of the vectors **A** and **B**. The magnitude of the vector $\mathbf{R}=\mathbf{A}\times\mathbf{B}$ is then;

$$\boxed{|\mathbf{R}|=\sqrt{(A_yB_z-A_zB_y)^2+(A_xB_z-A_zB_x)^2+(A_xB_y-A_yB_x)^2}} \tag{2.31}$$

2.6 Problems

1. An airplane is traveling due north-east at 800 km/h. Find the components of its velocity along the East and the North.
2. A person walks to the East for 2 km and then toward the North for 1 km. Calculate the resultant displacement vector and its angle with the East.
3. A worker ant carrying a load twice her size is zigzagging her way home. If she makes twenty zigzags to reach home and assuming each zigzag is a right triangle where the sides make a 45° angle with the vector toward home, calculate the ant's distance to her home.
4. Use the parallelogram rule to find the magnitude and the direction of the resultant vector for two displacement vectors making a 30° angle with each other and having magnitudes of 10 m and 15 m.
5. Find the difference of the two vectors described in problem 4.
6. A mass with weight of 100 N is resting on a 30° inclined plane. Find the component of the weight along the plane and perpendicular to the inclined plane.
7. Three forces, $\mathbf{F_1}=5\ N$, $\mathbf{F_2}=8\ N$ and $\mathbf{F_3}=10\ N$ are applied at a point O and making angles of 10°, 80° and 230°, respectively with the x-axis. Using the component method find the resultant force.

8. The resultant force of three forces is 500 N in the Northeast direction. Two of the forces are 100 N due West-Southwest and 200 N due South. Find the magnitude and the direction of the third force, **F**.

9. Find the sum and the difference of the two vectors $\mathbf{A} = 5\mathbf{i} + 10\mathbf{j}$ and $\mathbf{B} = 3\mathbf{i} - 7\mathbf{j}$.

10. Find the sum and the difference of the two vectors $\mathbf{A} = 3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$.

11. Find the magnitude and the direction of the two vectors in problem 9.

12. Find the magnitude and the direction of the two vectors in problem 10.

13. A force of 30 N is pulling a cart at an angle of 30°. If the displacement of the cart is 500 m, find the work done by this force.

14. Find the dot product of the two vectors described in problem 9.

15. Find the dot product of the two vectors described in problem 10.

16. Torque is defined as the cross product of two vectors **r** (distance) and **F** (force) and it is $\tau = \mathbf{r} \times \mathbf{F}$. To loosen a lug nut on a wheel, we exert about 2000 N of force; if the $r = 1.0\,cm$, find the applied torque.

17. Find the cross product of the two vectors described in problem 9.

18. Find the cross product of the two vectors described in problem 10.

Chapter 3

Motion in One Dimension

The laws of physics are inspired by observation of the physical world. A quick glance immediately reveals that everything around us is moving and therefore we must describe their motion with physics tools at our disposal. Even objects that seem motionless, such as the Sun, are in fact moving around the center of our galaxy, the Milky Way, at approximately 250 km/s.

In order to describe motion, we need three different physical quantities, position, velocity and acceleration.

3.1 Position and Displacement

The position of an object simply refers to its location with respect to a xyz coordinate system. The position, or displacement, is a vector quantity having both magnitude and direction. The unit for the position vector in the SI system is *meter*. The position vector is simply the xyz coordinate with respect to a global or local coordinate system. Displacement is the change in the position of an object. For example, an object can change its position from (x_1, y_1, z_1) to (x_2, y_2, z_2). Since we are considering motion in one dimension, we can therefore confine all positions and displacements to only the x-axis. According to the above definition for displacement, position can be thought of as an object's location from the origin $(0, 0, 0)$.

3.2 Velocity

The rate of change of the position vector is velocity, denoted by the Latin letter v. Therefore, velocity tells us how fast an object is moving. If we have the position as a function of time, then we can obtain the velocity by simply taking the derivative of the position function with respect to time.

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \tag{3.1}$$

In equation 3.1, $\mathbf{v}$ is called the instantaneous velocity. If we express the position as $x = f(t)$, then the velocity is simply the derivative of this function with respect to time, meaning that the velocity vector is the tangent to the position curve.

According to equation 3.1, we can define the average velocity simply as the total displacement vector divided by the time it takes to complete the motion. Note again that the velocity is a vector as it is denoted by the bold-faced $\mathbf{v}$.

$$\mathbf{v} = \frac{\Delta \mathbf{x}}{\Delta t} \tag{3.2}$$

3.3 Acceleration and Deceleration

The rate of increase in velocity is acceleration and the rate of decrease in velocity is deceleration. Therefore, if the velocity does not change, then the acceleration or deceleration is zero. By this definition, we can express acceleration $\mathbf{a}$, which again is a vector, as;

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}. \tag{3.3}$$

We can also express the acceleration as the second derivative of position with respect to time.

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}. \tag{3.4}$$

If we assume that acceleration is constant, then we can derive the functional dependence of both velocity and position mathematically by using equation 3.3.

$$d\mathbf{v} = \mathbf{a}dt \tag{3.5}$$

Now, in order to find $\mathbf{v}$, we must integrate equation 3.5 with respect to time.

$$\mathbf{v} = \int \mathbf{a}dt \qquad\qquad \mathbf{v} = \mathbf{a}t + C \tag{3.6}$$

In equation 3.6, C is the constant of integration which we replace with v_0 denoting the initial velocity of the object. Therefore, we can express velocity as a function of time as;

$$\boxed{\mathbf{v} = \mathbf{a}t + v_0}. \tag{3.7}$$

We know from equation 3.1 that velocity is the derivative of position with respect to time. Therefore, if we integrate equation 3.7 with respect to time, we obtain the functional dependence of the position vector.

$$\mathbf{x} = \int \mathbf{a}t + v_0 dt \tag{3.8}$$

$$\mathbf{x} = \frac{1}{2}\mathbf{a}t^2 + v_0 t + C \tag{3.9}$$

In equation 3.9, C, again, refers to the constant of integration and we therefore replace it with x_0, denoting the initial position of the object. We then can write the position vector as;

$$\boxed{\mathbf{x} = \frac{1}{2}\mathbf{a}t^2 + v_0 t + x_0} \tag{3.10}$$

In many cases, however, we usually choose our coordinate systems such that $x_0 = 0$.

Example 1. A car is accelerating from rest to 36.0 km/hr in 5.0 s. Calculate the acceleration of the car.

Answer:

First we must convert the velocity from km/hr to m/s.

$$v = \frac{36.0 \times 1000}{3600} = 10.0\ m/s \tag{3.11}$$

Using equation 3.7, we can write;

$$a = 10.0/5.0, \tag{3.12}$$

or;

$$\boxed{a = 2.0\ m/s^2} \tag{3.13}$$

3.4 Time Independent Relations

Now that we have derived the position and the velocity as a function of time, we would like to find a relation between position, velocity and acceleration. To achieve this, we must eliminate time in equations 3.7 and 3.10. We therefore calculate t from equation 3.7 in terms of v, a, and x and substitute it in equation 3.10. Note, because the motion is restricted to only one dimension, we can remove the bold-faced notations.

$$t = \frac{v - v_0}{a} \tag{3.14}$$

$$x = \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\frac{v - v_0}{a} + x_0 \tag{3.15}$$

$$x - x_0 = \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\frac{v - v_0}{a}. \tag{3.16}$$

Now, equation 3.16 is a time independent relation expressing position as a function of velocity and acceleration. At this stage, the physics part is over and further simplification of equation 3.16 is just algebra. Let us proceed.

$$x - x_0 = \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\frac{v - v_0}{a}. \tag{3.17}$$

Further simplifications yield;

$$x - x_0 = \frac{v^2 + {v_0}^2}{2a} - \frac{vv_0}{a} + \frac{vv_0}{a} - \frac{{v_0}^2}{a} \tag{3.18}$$

The second and third term in 3.18 cancel out and combining the last term with the second part of the first term yields $-\frac{{v_0}^2}{a}$. We therefore can write;

$$\boxed{x - x_0 = \frac{v^2 - v_0^2}{2a}} \tag{3.19}$$

Example 2. A car with a constant velocity of 10.0 m/s brakes for a red light. If the brake deceleration is 5.0 m/s^2, calculate the distance the car would travel before coming to a stop and the time.

Answer:

We use equation 3.19 and plugging in the values for v_0 and a, we have;

$$x = \frac{10.0^2}{2 \times 5.0}, \tag{3.20}$$

or;

$$\boxed{x = 10.0\ m}. \tag{3.21}$$

The time is;

$$t = \frac{v_0}{a}. \tag{3.22}$$

Plugging in the values for v_0 and a;

$$\boxed{t = 2.0\ s} \tag{3.23}$$

3.5 Free Fall in Vacuum

An excellent example of motion in one dimension is free fall in vacuum. We may consider this motion as a constant acceleration motion. This can be achieved when an object is near a massive body such as the Earth or the Moon. The constant acceleration is a vector that always points downward and is referred to as the acceleration of gravity, denoted by the letter **g**.

Equations of motion remain identical to equations describing position and velocity derived above. Keep in mind that acceleration **a** is now replaced with $-g$. The minus sign points to the fact that the acceleration of gravity vector is pointing downward. We can therefore write the equation for the position as a function of time;

$$\boxed{\mathbf{y} = \frac{-1}{2}\mathbf{g}t^2 + v_0 t + y_0} \tag{3.24}$$

Note we also denoted the position with y instead of x to signify that the motion is in the vertical direction rather than the horizontal direction.

Similarly, the velocity as a function of time can be written as;

$$\boxed{\mathbf{v} = -\mathbf{g}t + v_0} \tag{3.25}$$

The time-independent equation can be written according to equation 3.19 as;

$$\boxed{y - y_0 = \frac{v^2 - {v_0}^2}{-2g}} \tag{3.26}$$

Note, as the title of this section implies, these equations hold true only in vacuum. The presence of air or any other drag force would change the form of these equations and make them more complicated.

Example 3. A ball is thrown vertically upward and it reaches a height of 5.0 m. Calculate the initial velocity of the ball.

Answer:

Using equation 3.26 we can write;

$$y = \frac{v^2}{2g}, \tag{3.27}$$

or;

$$v = \sqrt{2gy}. \tag{3.28}$$

plugging values for y and g, we then have,

$$v = \sqrt{2 \times 9.8 \times 5.0}, \tag{3.29}$$

or;

$$\boxed{v \approx 9.9\ m/s} \tag{3.30}$$

3.6 Problems

1. A car is traveling at a constant velocity of 60.0 km/h. How far would it travel in 163 $minutes$?
2. A long distance runner running a 5.0 km run is pacing himself by running 4.5 km at 9.0 km/h and the rest at 12.5 km/h. What is his average speed?
3. Two trucks are traveling from two different cities towards the same destination city. If the distance of one truck is 2.5 times that of the other one, how fast do the two trucks have to travel to get to the destination city at the same time?
4. How long does it take to walk from the Earth to the Moon at an average speed of 5.0 km/h? Take the Earth-Moon distance to be 380000 km. Repeat the calculation for a jet traveling at 950.0 km/h.
5. Two jets, on the London-New York route are traveling in the opposite directions. If the head wind for the New York - London jet is 100 km/h, assuming the air speed of each jet is 900 km/h, calculate at what point along their route they would pass each other. Express you answer in terms of a distance ratio.
6. Two ships are traveling towards each other at 54 km/h and are initially 20 km apart. If they monitor each other by sonar and use one ping to locate the other one, how long does it take for the echo to reach the ship initiating the ping? Take the speed of sound in water to be 1424 m/s.
7. A dragster has a constant acceleration of 20 m/s^2. If the length of the track is 150 m, what is its velocity at the finish? How long does it take to reach the finish line?

8. A car is approaching a traffic light at a constant speed of 15 m/s. The driver sees the light turn yellow and slows down by applying the brakes. If he is 50 m away from the light when he applies the brakes, calculate his brake deceleration if he is to come to a complete stop at the light.

9. A ball is thrown vertically upward and reaches a height of 20 m. Find the initial velocity of the ball. Assume $g = 9.8\ m/s^2$. If the ball reaches the same height on the Moon, find the initial required velocity. ($g_{Moon} = g/6$).

10. A balloon at an initial height of 20 m is ascending at a constant speed of 3.0 m/s. If a box suddenly falls over and crashes to the ground, how long does it take for the balloon's occupants to hear the sound? Assume $g = 9.8\ m/s^2$ and $v_{sound} = 340\ m/s$.

11. A linear accelerator (Linac) is a device through which charged elementary particles such as electrons or protons are accelerated for various research goals and applications. The acceleration is achieved by the means of radio frequency controlled drift tubes where the sign of the electric field alternates to attract and repel the charged particle, i.e., give it a kick. If the electric field imparts an acceleration of $2.0 \times 10^{14}\ m/s^2$ to, say an electron, calculate its velocity after it travels 10 m. How long does it take for the electron to move this distance?

12. At what distance above the Earth and the Moon does an object take the same amount of time to reach the ground? ($g_{Moon} = g/6$).

13. The velocity of an object when it hits the ground is 10.0 m/s. From what height was it dropped? Find the time it took the object to hit the ground.

14. An object is dropped with an initial velocity of 10.0 m/s and it takes 2.0 s to hit the ground. Find the height at which it was released. Calculate its velocity when it hits the ground.

15. A sounding rocket is fired vertically at a constant acceleration of 20 m/s^2 and then the fuel is used up at the one minute mark. Calculate how high the rocket would climb.

16. A ball is thrown vertically upward from a balcony at a height of 2.5 m above the street. If the time for the ball to hit the street is 3 s, find the initial upward velocity of the ball.

17. A rock is thrown vertically upward with an initial velocity of 2.0 m/s. If it hits the ground with a velocity of 10.0 m/s, calculate the height from which it was thrown.

18. In problem 17, calculate the time it takes the rock to hit the ground.

19. A car moving with an initial constant velocity covers a 50 m AB span of a highway in 5.0 s. If its velocity at point A is 15.0 m/s, what is its velocity as it passes point B?

20. In problem 19, find the acceleration of the car. Calculate how far behind point A the car started to move.

Chapter 4

Motion in Two and Three Dimensions

In Chapter 3, we described the motion of an object in one dimension. Motion in one dimension constrains kinematics vectors such as position, velocity and acceleration to a straight line. We, however, live in a three dimensional world and an object usually is not constrained to one dimension when it moves. Therefore, the description of vectors such as position, velocity and acceleration must take the form of a general two or three dimensional vector.

4.1 Position and Displacement

The position of an object in two or three dimensions simply refers to a vector showing its location with respect to a xyz Cartesian or r, θ, ϕ spherical coordinate system or any other coordinate system. As we mentioned in chapter 3, the position or displacement is a vector quantity having both magnitude and direction. The following vector equations represent the position vector as a function of its components in a xyz Cartesian coordinate system.

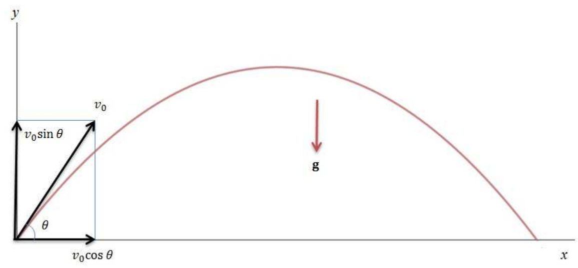

Figure 4.1: Schematic view of a projectile motion in vacuum.

$$\boxed{\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}} \tag{4.1}$$

4.2 Motion in a Plane

In chapter 3 we discussed the motion of an object in one dimension, but we will now generalize the problem in two dimensions. This is the classic projectile motion in a vacuum medium. We stress the vacuum medium to point to the fact that if air resistance is involved then equations are more complicated and we shall address this problem later in this text.

As shown in figure 4.1, the object is thrown at any angle θ with an initial velocity v_0.

Note the only acceleration acting on the projectile is the acceleration of gravity $\mathbf{g}$ in the vertical or y direction. In the horizontal or the x direction, the motion is uniform. Let us develop the equations of motion.

$$v_x = v_0 \cos\theta \tag{4.2}$$

$$v_y = v_0 \sin\theta. \tag{4.3}$$

Now that we have resolved the initial velocity vector $\mathbf{v_0}$ in the x and the y directions, we can describe the motion in these two dimensions.

As stated above, the motion in the $x - direction$ is uniform, therefore, we can write;

$$x = v_0 t \cos\theta. \tag{4.4}$$

In the $y - direction$, the motion is uniformly accelerated due to the acceleration of gravity. We can therefore write;

$$y = -\frac{1}{2}gt^2 + v_0 t \sin\theta. \tag{4.5}$$

From equation 4.5, we can find the velocity equation by simply taking the derivative with respect to time;

$$v_y = -gt + v_0 \sin\theta. \tag{4.6}$$

Example 1. Find the vertical position and the velocity of a ball which is thrown at an angle of $\theta = 30°$ with an initial velocity of 10.0 m/s at $t = 1.0$ s.

Answer:

We use equation 4.5;

$$y = -\frac{1}{2}9.8(1.0)^2 + (10.0)(1.0)\sin 30°. \tag{4.7}$$

Then;

$$\boxed{y = 0.1\ m}. \tag{4.8}$$

And using equation 4.6, we find the velocity;

$$v_y = -9.8(1.0) + 10.0 \sin 30°, \tag{4.9}$$

or;

$$\boxed{v_y = -4.8\ m/s}. \tag{4.10}$$

4.2.1 Equation of the Trajectory

In figure 4.1 we see the path of the projectile in red. We will show that this trajectory is a parabola. We will do this by eliminating the time (t) between the two equations 4.4 and 4.5. This is achieved by calculating the time t from equation 4.4 in terms of x and v_0 and substituting it in equation 4.5.

$$t = \frac{x}{v_0 \cos\theta} \tag{4.11}$$

Now substituting for t in equation 4.5 we obtain the equation for the particle trajectory.

$$\boxed{y = -\frac{gx^2}{v_0^2 \cos^2\theta} + x\tan\theta} \tag{4.12}$$

A careful examination of equation 4.12 reveals that it is of the form $y = ax^2 + bx$ which is the equation of a parabola.

4.2.2 Projectile Range

Range of the projectile is defined as the distance it travels before it hits the ground. Equation 4.12 is very useful for finding the range of the projectile. To obtain the range R, all we have to do is to solve for x when $y = 0$.

$$\frac{gx^2}{v_0^2 \cos^2\theta} + x\tan\theta = 0 \tag{4.13}$$

One solution to equation 4.13 is obviously 0 and that is where the object is initially thrown. The other value of x is R or the range.

$$\frac{gx}{v_0^2 \cos^2\theta} = \tan\theta, \tag{4.14}$$

or,

$$\frac{gx}{v_0^2 \cos^2\theta} = \frac{\sin\theta}{\cos\theta} \tag{4.15}$$

$$x = \frac{v_0^2 \sin\theta\cos\theta}{g}. \tag{4.16}$$

Using the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$ we can write;

$$\boxed{R = \frac{v_0^2 \sin 2\theta}{2g}}. \tag{4.17}$$

Example 2. Find the projectile angle which yields the longest range.

Answer:

We start with the range equation 4.17 and we will take the derivative of R with respect to the angle θ.

$$\frac{dR}{d\theta} = \frac{v_0^2}{g}(\cos 2\theta). \tag{4.18}$$

To maximize R, we must set $\frac{dR}{d\theta}$ equal to zero.

$$\frac{v_0^2}{g}(\cos 2\theta) = 0, \tag{4.19}$$

or,

$$\cos 2\theta = 0, \tag{4.20}$$

and;

$$2\theta = 90^\circ, \tag{4.21}$$

or;

$$\boxed{\theta = 45^\circ}. \tag{4.22}$$

4.3 Circular Motion

The motion of a particle along a circular path is defined by its velocity and its acceleration. The velocity is always tangent to the displacement curve, in this case, the circle. We know from chapter 3 that the velocity is a vector and although its magnitude is a constant, its direction constantly changes as it moves on the circle. This change is what causes the acceleration. This acceleration is referred to as the "centripetal" acceleration. The word centripetal means seeking a center. The acceleration therefore always points radially inwards towards the center of rotation. Figure 4.2 shows the velocity and the acceleration vectors.

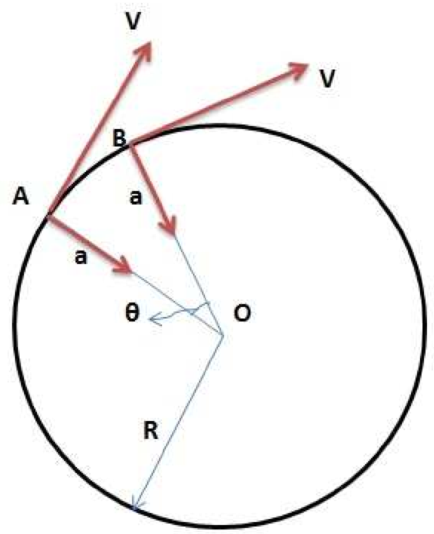

Figure 4.2: Velocity and acceleration for a particle moving on a circle from point A to point B. Note that magnitude of the velocity is a constant, however, its direction is constantly changing.

Now we shall derive the centripetal acceleration in terms of the velocity and the radius R. Let us assume that the displacement in figure 4.2 shown as AB is infinitesimal and we denote it as ds. We can also then assume the angular displacement θ is also infinitesimal and we can denote it as $d\theta$. From geometry we can also deduce that the change in the velocity vector is dv and its angular change is also $d\theta$ since v is perpendicular to R. We can therefore write;

$$ds = Rd\theta; dv = vd\theta. \tag{4.23}$$

Also since the magnitude of the velocity is a constant, $ds = vdt$, and substituting from equation 4.23 for ds; $Rd\theta = vdt$. However from equation 4.23 $d\theta = dv/v$. We therefore can write;

$$v = \frac{Rdv}{vdt}, \tag{4.24}$$

however, $a = dv/dt$;

$$\boxed{a = \frac{v^2}{R}}. \tag{4.25}$$

4.4 Frequency and Period

The repetitive nature of the uniform circular motion requires the definition of two related concepts, namely the frequency and the period. The frequency, denoted with f, is defined as the number of times per unit time that the same state or configuration of a system is repeated. The units of frequency is cycle per second or *Hz*.

The period, denoted with T, is defined as the time it takes for the particle or the object to complete one full cycle. Therefore, the mathematical relation between the period and the frequency is;

$$f = \frac{1}{T}. \tag{4.26}$$

The unit of the period is *second*.

Example 3. The period of Phobos, the larger of the two Martian natural satellites is 7.66 h. If the distance of Phobos is 9377 km from the center of Mars, find the centripetal acceleration experienced by the moon.

Answer:

From the period and the radius given above we find the velocity of Phobos.

$$T = 7.66 = 27576\ s, \tag{4.27}$$

and,

$$R = 9377 = 9377000\ m. \tag{4.28}$$

Now we find the circumference of the orbit.

$$C = 2\pi R = 58917566.4\ m. \tag{4.29}$$

The orbital velocity of Phobos then is;

$$v = \frac{C}{T} = \frac{58917566.4\,m}{27576\,s} \approx 2136.6\,m/s. \tag{4.30}$$

We can now calculate the centripetal acceleration.

$$\boxed{a = \frac{v^2}{R} \approx 0.487\ m/s^2}. \tag{4.31}$$

4.5 Problems

1. A ball is thrown at a 45° angle and achieves a range of 40 m. Calculate the initial velocity of the ball.
2. A projectile has an initial velocity of 10.0 m/s and reaches a maximum height of 5.0 m. Calculate the angle of the projectile.
3. The maximum range of a ball is 50 m. What is the maximum range on the Moon? ($g_{Moon} = g/6$).
4. In a game of American football, a ball is kicked in a field goal attempt at an angle of 40°. It has to clear the cross bar 50.0 m down field at a height of 3.1 m. What should be the initial velocity of the football?
5. A basketball is thrown 20 m down court at an angle of 30° and initial velocity of 10 m/s. If a player is 5 m away from the ball, how fast should he run in order to catch the ball before it hits the court? Assume that the players are the same height.
6. An electron horizontally enters a meter-long drift tube, i.e., no acceleration, at an initial velocity of 30, 000 km/s. Calculate how far it drops down due to gravity as it exits the drift tube.
7. In an Olympic shooting contest, a shooter is trying to hit a target 100 m away at the same level of the rifle. If the muzzle velocity is 600 m/s, calculate how far above the target the shooter should hold the rifle in order to hit the target.
8. A small meteor is falling on Earth at a constant velocity of 1000 km/h. We want to blow the meteor out of the sky before it reaches the ground. A ground defense can fire a radar controlled projectile at the meteor with the velocity of 2000 km/h. If the meteor is 10 km above the Earth and

falling vertically and the air defense system is 10 km away, calculate the requirements for collision of the two objects, i.e., hitting a bullet with a bullet! (Hint: find the time and the angle at which the projectile has to be fired.)

9. In the Hydrogen atom, an electron is assumed to revolve around the proton at a speed of $\approx 2.24 \times 10^6$ m/s in a circular orbit. Calculate the acceleration of the electron. The radius of the H-atom is $0.5\mathring{A}$.

10. One way to produce artificial gravity on deep space voyages is to make the spacecraft rotate about a given axis. Assume the craft is a cylinder with base radius of 10.0 m. At what angular velocity should the spacecraft rotate to produce a gravity equal to that of the Earth? Also do the calculation for Mars and the Moon.

11. The acceleration of gravity of the Sun at the orbit of Mars is 0.00257 m/s^2. If the radius of the orbit of Mars is 228 million km, calculate the orbital velocity of Mars.

12. It is suggested that a very young Earth was spinning so fast that its days only lasted 2.5 h instead of today's 24 h. Calculate the acceleration of gravity on the equator in the young Earth.

13. A H-atom near a neutron star has its electron removed by a strong electric field. If the speed of the electron is $\approx 2.24 \times 10^6$ m/s and is making an angle of $60°$ with the horizon and the field of gravity of the neutron star is perpendicular to the horizon and its range is 32.0 cm, what is the acceleration of gravity of the neutron star?

14. A near Earth satellite has an orbital period of 100 minutes at a height of 650 km. If the radius of Earth is 6400 km, find the acceleration of gravity experienced by the satellite.

15. Find the height above the Earth for a satellite orbit where the period is 24 h. Note this is called the geostationary orbit.

16. A plane propeller has a frequency of 2400 RPM. If the radius of the propeller is 75.0 cm, find the velocity of the tip of the propeller.

17. A mass attached to a string with the length R is rotating in a vertical plane. Find the velocity of the rotation as a function of R and g such that at the highest point the net acceleration of the mass is zero.

18. Galactic motion of the stars seems to be constant as a function of their distance to the galactic center. This contradicts the laws governing the velocities of the planets around the Sun. Find the condition forcing the velocity of the Sun at 8 kpc and another star at 20 kpc to be the same.

19. If the orbital velocity of the Sun around the galaxy is 230 km/s and the Sun is approximately 8 kpc from the center of the galaxy, find the orbital period of the Sun, its frequency and angular velocity.

20. A mass is attached to a meter-long string and is rotating in a vertical plane. When it reaches the highest point it has a net acceleration of g and the string fails, calculate the range of the mass as it flies into the air. Assume the highest point is 2.5 m from the ground.

Chapter 5

Force and Dynamics

In Chapters 3 and 4 we discussed the kinematics of a particle but we did not discuss the cause for the motion or how its mass actually contributes to its motion. Although we discussed a projectile motion in the field of gravity, we avoided introducing the notion of force due to gravity. In this chapter, we will define mass and then we will define the vector quantity force causing the motion. The formal treatment of particle dynamics that we know today as classical mechanics, is due to Sir Isaac Newton (1642-1727) and the equations or the laws are referred to as Newton's Laws of motion and the mechanics is called Newtonian Mechanics. Newton's Laws of motions are traditionally expressed as three laws and will be discussed in the following sections.

Before we move on to studying Newton's Laws of motion, we define mass. We believe the definition of mass at this juncture is essential to a better understanding of Newtonian Mechanics.

Definition: *Mass is a property of matter that gives rise to the ability of a body to resist acceleration.*

This definition seems very intuitive because we experience it in our everyday life. We expect and observe that smaller and less massive objects accelerate much quicker. For example, a sports car accelerates a lot faster than a dump truck. A small person can run faster than a large person and so on.

5.1 Newton's First Law

Newton's First Law of motion states that if the sum of forces on an object is zero, then the object is either at rest or is moving uniformly. The implication of Newton's First Law is that the acceleration of the object is zero. Mathematically we can write this law as;

$$\boxed{\Sigma \mathbf{F} = 0}. \tag{5.1}$$

For example, household items sitting around the house such as chairs or tables are stationary because the sum of all forces on them is zero.

5.2 Newton's Second Law

Newton's Second Law of motion states that net force applied on an object is directly proportional to its acceleration with the proportionality constant m, the mass of the object. Mathematically we can write Newton's Second Law as;

$$\boxed{\Sigma \mathbf{F} = m\mathbf{a}}. \tag{5.2}$$

Note, this definition is only valid for a constant mass. In more complex systems such as cars, planes and rockets the mass is a variable due to fuel consumption by the system. We will redefine Newton's Second Law later in the text when we introduce the concept of *momentum* ($\mathbf{p}$).

Note, equation 5.2 can be written as $\Sigma \mathbf{F} = m\frac{d\mathbf{v}}{dt}$. We recognize this as a first order ordinary differential equation. We can further write $\Sigma \mathbf{F} = m\frac{d^2\mathbf{x}}{dt2}$, and this is a second order ordinary differential equation. We will use this formalism later to develop the concept of *momentum* and *impulse*.

Newton's Second Law of motion is the cornerstone of Newtonian Mechanics and many laws of nature can be derived from this simple equation.

5.3 Newton's Third Law

This is the law of action-reaction. It states that for every action, there is an equal and opposite reaction. This law can be understood intuitively. If we put an object on the floor, the reason it stays there is because the force exerted by the object on the floor, i.e., action, is equal and opposite to the force exerted on the object by the floor, i.e., reaction.

Isaac Newton first compiled these three laws in his *PhilosophiæNaturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), which was published in 1687. He used them to investigate and explain the dynamics of physical objects and systems. In the third volume of the text, Newton showed that these laws when combined with his law of universal gravitation explained Kepler's Laws of planetary motion (1609).

5.4 Weight

Many people have a misconception of weight. For example, when asked how much they weigh, someone might answer 85 kg! However, we know that kg is a unit of mass, hence 85 kg simply refers to the mass of the person which remains unchanged no matter whether here on Earth or any other place in the universe. Weight, however, is a force (a vector) and can be defined by Newton's Second Law when **a** is replaced with **g**. We therefore can write;

$$\boxed{\mathbf{W} = m\mathbf{g}} \tag{5.3}$$

Because weight is a function of acceleration of gravity **g**, it varies depending on the magnitude of **g**. The reason astronauts can hop on the Moon is because **g** has only a magnitude of 1.6249 m/s^2 instead of the 9.81m/s^2 here on Earth. So a person on the Moon feels only 16.7% of what he/she weighs here on Earth. We also feel this effect here on Earth when we are in a fast elevator going down or in an amusement park on a roller coaster. We illustrate this physical observation in the following example.

Example 1. An elevator is moving down with a constant acceleration of 1.00 m/s^2. How much lighter does an $80-kg$ man feel riding the elevator.

Answer:

First we find the weight of the man.

$$W = mg; W = 80 \times 9.81$$
$$W = 784.8\ N$$

Now we calculate the weight of the man in the elevator.

$$W' = m(g-a); 80 \times (9.81 - 1.00)$$
$$W' = 704.8\ N$$

We are almost there, let us subtract W' from W and we have;

$$\Delta W = W - W' = 784.8 - 704.8$$
$$\boxed{\Delta W = 80\ N}$$

Hence, the man feels 80 N lighter when traveling downward in the elevator.

5.5 Elastic Force

Elastic force is the force which arises from the deformation of a solid such as a spring or a rubber band. The elastic force is proportional to the deformation of the object through the following equation;

$$\mathbf{F} = -k\mathbf{x} \tag{5.4}$$

Equation 5.4 is known as Hooke's law. In this equation k is called the spring constant. The spring constant is a property of the spring and a measure of its "stiffness". The minus sign indicates the fact that the direction of the pull (push) of the spring is always opposite to that of the applied force.

Example 2. A block is resting on a spring with $k = 1000.0\ N/m$. If the spring is compressed by $x = 2.0\ cm$, calculate the mass of the block.

Answer:

First we find the elastic force in the spring using equation 5.4 and keeping in mind that $2.0\ cm = 0.02\ m$.

$$F = 1000 \times 0.02 = 20.0\ N \tag{5.5}$$

This force of $20.0\ N$ is the weight of the block or $W = mg$. Therefore, the mass is;

$$\boxed{m = \frac{20.0}{9.8} \approx 2.041\ kg}. \tag{5.6}$$

5.6 Friction and Dissipative Forces

Frictional forces are retarding forces which oppose the motion of an object. This can be shown mathematically as;

$$\mathbf{f} = \mu \mathbf{N} \tag{5.7}$$

In equation 5.7 μ is the coefficient of friction and it is a property of material or the substance. It is actually a measure of the "roughness" of a given surface. For example, we expect and rightly so that it is easier to move a piece of ice on a glass surface than to move a rock on a gravel road. There are two types of friction, *static* friction and *kinetic* friction. We describe these two forces in subsections below.

5.6.1 Static Friction

The static friction is the frictional force which exists when two objects are at the verge of motion. The force is;

$$\mathbf{f_s} = \mu_s \mathbf{N}. \tag{5.8}$$

The μ_s is called the coefficient of static friction and again it is a property of material. We can actually measure μ_s by a very simple experiment.

Assume you want to measure the coefficient of static friction for a block on a given surface. Take a board and lay an object on it then simply raise the end of the board slowly until the object is just about to slide. Then measure the angle of the incline and take its tangent and we will prove below that this is the coefficient of static friction, μ_s.

The object's weight is;

$$w = mg, \tag{5.9}$$

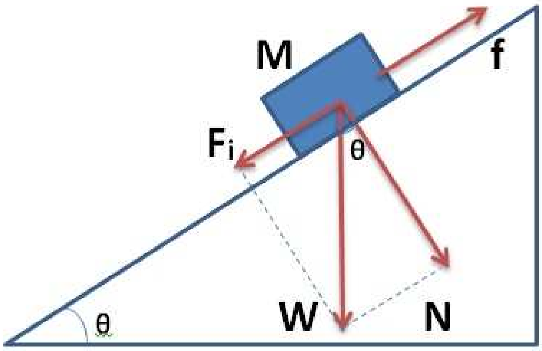

Figure 5.1: The forces acting on a block of mass M about to slide on a rough inclined plane.

and as shown in the figure 5.1 W has two components, one normal to the inclined plane N and the other along and parallel to the inclined plane F_i. We can therefore write;

$$N = mg\cos\theta, \tag{5.10}$$

and for F_i;

$$F_i = mg\sin\theta. \tag{5.11}$$

Note the force F_i is the one responsible for driving the object down the incline. Now we have to calculate the force of friction trying to hold the object in place on the incline. According to equation 5.8;

$$f_s = \mu_s N = \mu_s mg\cos\theta, \tag{5.12}$$

at the instant when the object starts slipping on the inclined plane we must have $F_i = f_s$. We can therefore write;

$$mg\sin\theta = \mu_s mg\cos\theta. \tag{5.13}$$

Solving for μ_s in equation 5.13, we note m and g cancel out and we know from trigonometry that $\tan\theta = \sin\theta/\cos\theta$. Hence, we have;

$$\boxed{\mu_s = \tan\theta}. \tag{5.14}$$

5.7 Dynamics of Uniform Circular Motion, The Centripetal Force

In chapter 4 we defined and derived the centripetal acceleration as $a = v^2/r$. In this chapter we discuss the force created by this acceleration from Newton's Second Law.

$$\boxed{F_c = \frac{mv^2}{r}} \tag{5.15}$$

Note the direction of the force F_c is along the centripetal acceleration and points inwards toward the center of rotation.

It seems appropriate at this juncture to introduce angular velocity ω. From geometry we know the relation between distance or arc length s, the angle, θ in radians and the radius r is;

$$s = r\theta. \tag{5.16}$$

If we take the derivative of both sides of 5.16 with respect to time we then can write;

$$\frac{ds}{dt} = r\frac{d\theta}{dt}. \tag{5.17}$$

In equation 5.17 $\frac{ds}{dt}$ is the velocity v and $\frac{d\theta}{dt}$ is the rate of change of angle θ with time and therefore angular velocity ω. We then rewrite equation 5.15 as

$$\boxed{F_c = mr\omega^2}. \tag{5.18}$$

Example 3. The orbital velocity of Mars is 24.1 km/s. If the average distance of Mars from the Sun is 228000000 km and its mass is 0.11 that of the Earth, find the centripetal force experienced by Mars.

Answer:

Mass of the Earth is approximately 5.96×10^{24} kg and therefore the mass of Mars is;

$$M_{Mars} = 0.11 \times 5.96 \times 10^{24}\ kg \approx 6.6 \times 10^{23}\ kg \tag{5.19}$$

Using equation 5.15 and keeping in mind that $1.0\ km = 1000\ m$ we can write;

$$F_c = \frac{6.6 \times 10^{23} \times 24100^2}{2.28 \times 10^{11}}, \tag{5.20}$$

or;

$$\boxed{F_c = 1.68 \times 10^{21}\ N}. \tag{5.21}$$

5.8 Problems

1. A block with mass m is sliding on a smooth surface with velocity v_0 and then it reaches a 30° inclined plane. Calculate how far up the incline it would travel. Express your answer in terms v_0 and g.

2. A mass $m = 5.0\ kg$ is pressing against a spring on a horizontal surface and compresses it by 2 cm. If the mass attains an acceleration of 2.0 m/s^2, calculate the spring constant.

3. In problem 2, find an expression for the velocity of the block. Express your answer in terms of m, x, and k.

4. A mass is sliding from rest down a smooth inclined plane with an inclination angle θ. Calculate the velocity after it travels a distance l along the incline.

5. A block with mass m is held horizontally on a smooth surface against a spring with a spring constant k. If the deformation of the spring is x, and then we release the block, find its velocity as it detaches from the spring.

6. Repeat the problem number 5 when the spring-block assembly is in the vertical position.

7. Two blocks with masses M_1 and M_2 are squeezing a spring on a smooth horizontal surface. We then release both blocks, find the ratio of the acceleration of the two blocks.

8. In nuclear power plants thermal neutrons are captured by ^{235}U nuclei where nuclear fission produces heat. If the most probable velocity of thermal neutrons is about 2300 m/s and a thermal neutron gets captured on a ^{235}U nucleus with a diameter of 15 fm, calculate the braking force on the neutron. Look up the mass of the neutron on the web.

9. A block of mass is sliding up an inclined plane with an inclination angle θ. If the initial velocity of the block is v_0, find an expression for the distance l that the block would travel up the plane. Express your answer in terms of θ, v_0, and g.

10. In problem 9, if $v_0 = 10\ m/s$ and the length of the inclined plane is 5 m and the $\theta = 30°$, find the point at which the block hits the ground.

11. A sphere of electrical charge weighing 980 dyn is attached to a string which is attached to a wall having a like charge which repels the sphere. If the angle of the string with the vertical plane is 30°, find the tension in the string.

12. A locomotive has three similar cars with mass of 10 tons. If the force of the engine is 30000 N, find the tensions between cars 1 and 2, and 2 and 3. Neglect friction.

13. A block of mass m is on an inclined plane with an inclination angle θ. It is attached to a hanging mass M with nearly massless cord. Find the condition for the two masses where the whole system moves at a constant velocity. Find the tension in the cord.

14. A "governor" is a device for maintaining uniform speed in a machine, engine, etc., regardless of the fuel supply to the engine. This would insure an upper limit to the speed of the car. It is simply a pendulum device rising only to the horizontal level. If two masses of 250 grams each are attached to a 20.0 cm long arm symmetrically, what is the maximum angular velocity of the drive shaft?

15. A block of mass $M = 5.0\ kg$ is on a rough surface moving with constant velocity. If the coefficient of dynamic friction is 0.1, find the necessary force required for this steady motion.

16. A block of mass m is resting on the top of a larger mass M on a horizontal smooth surface. A force $\mathbf{F}$ along the horizon is pulling on M. What should be the minimum coefficient of static friction between m and M so that m would not slide off M?

17. In amusement parks people have fun by going into a device called the *rotor*. This device is a large cylinder which rotates on an axis and people stand on the bottom of the tank with their backs to the cylinder's wall. When

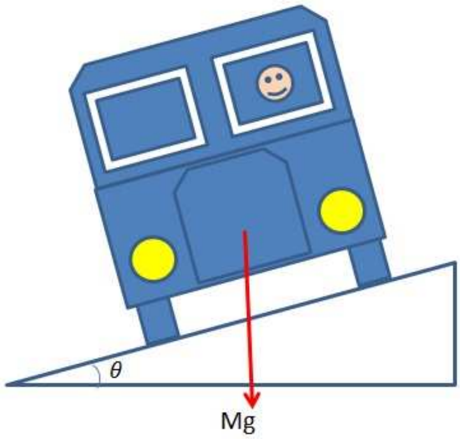

Figure 5.2: A car driving around the curve of a highway which is banked by an angle $\theta = 20°$.

the rotation achieves a certain angular velocity the bottom drops down but people are still stuck to the wall. If the coefficient of static friction is μ, find the minimum angular velocity of the cylinder that insures a person stays attached to the wall.

18. A civil engineer is given the task to determine the speed limit on a highway curve with a 40 m radius where there is a designed bank of $\theta = 20°$ as shown in figure 5.2. If the coefficient of static friction between the tires and the road is 0.15, what would the engineer calculate for the maximum speed of a car before it loses control and flips over?

19. An $80 - kg$ man on a $10 - m$ ladder is half way up. The ladder is resting on a wall at an angle of 30°. What is the coefficient of static friction between the ladder and the wall or the ground that prevents the ladder from slipping?

20. The average acceleration of gravity on Jupiter is about 27 m/s^2. If the day on Jupiter is approximately 10 hours, how much would an object weigh at the equator?

Chapter 6

Work, Energy and Conservation of Energy

In Chapter 5 we discussed Newton's Laws of motion and the concept of force where we dealt with vectors to obtain various properties of motion. In this chapter, we explore a branch of mechanics where we deal with work and energy in order to obtain a dynamical variable associated with motion of an object. Note, work or energy is a scalar quantity and we define it as;

Definition: *Work done by a constant force on an object is defined as the scalar product of the force and distance through which the object moves.*

Mathematically we can write work or W as;

$$\boxed{W = \mathbf{F} \cdot \mathbf{d}} \tag{6.1}$$

Note since the dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is $AB\cos\theta$, the work done by $\mathbf{F}$ on the object in equation 6.1 is maximum when $\mathbf{F}$ and $\mathbf{d}$ are parallel and zero when perpendicular.

The unit of work in the SI system is called *Joule* and it is denoted by the Latin letter J.

Energy is the same as work and is defined as the ability to do work. For example, an energetic person is able to get a lot of work done or a strong source of energy is a source that can perform a great deal of work and so on.

Often here on Earth, we do work in the field of gravity. In this case the work done in the horizontal direction has zero effect in the vertical direction where gravity is present. If we apply a constant force on an object with mass M in the horizontal direction with no resistance to the applied force such as friction, then the constant force causes the object to accelerate according to Newton's Second Law of motion, $\mathbf{F} = m\mathbf{a}$. This is exactly what happens when an object is in free fall due to its weight in the field of gravity. We use this as a segue to discuss two common energies, namely kinetic and potential energy.

6.1 Variable Force

If the force is not a constant magnitude force and it varies with distance, then we must include the functional dependence of the force and integrate the product to obtain the work done by the force. This statement has the following mathematical representation;

$$\boxed{dW = \int_1^2 F(x)dx}. \tag{6.2}$$

The limits on the integration are the initial and the final points of the path where the work is to be calculated. We shall see the application of equation 6.2 in the following section.

6.1.1 Kinetic Energy

As the name implies, kinetic energy is the energy associated with a moving object. For example, a moving projectile has kinetic energy and this is obvious when the projectile hits another object causing damage such as a pebble hitting the windshield of a car.

Using equation 6.2, we can now derive an equation for the kinetic energy as a function of mass and velocity.

$$dW = \int_1^2 F(x)dx \tag{6.3}$$

Substituting in equation 6.3 from Newton's Second Law $F = ma$; or, $F = mdv/dt$, we then have;

$$dW = \int_1^2 m\frac{dv}{dt}dx. \tag{6.4}$$

We also know from chapter 3 that $v = dx/dt$ or $dx = vdt$ and now substituting in equation 6.4 for dx;

$$dW = \int_1^2 m\frac{dv}{dt}vdt. \tag{6.5}$$

In equation 6.5 dt cancels out and we can also take the mass outside the integrand. We are then left with an integral equation in terms of velocity and the two end-points are simply the initial and the final velocities of the object or the particle. We, therefore, can write;

$$dW = m\int_{v_1}^{v_2} vdv. \tag{6.6}$$

Integrating equation 6.6 we get;

$$W = \frac{1}{2}m(v_2^2 - v_1^2). \tag{6.7}$$

We denote the kinetic energy with the Latin letter T and we can also conclude that at any moment a moving particle with a given velocity v carries the following kinetic energy;

$$\boxed{T = \frac{1}{2}mv^2}. \tag{6.8}$$

6.1.2 Potential Energy

The potential energy refers to the work done by the weight of an object against the force of gravity. For example if we raise an object to a height h above the ground it stores a potential energy equal to its weight times the aforementioned height h. Obviously, we did the same amount of work raising the object to the height h. We denote the potential energy with the Latin letter U and again just like work, its unit in the SI system is *Joule*. Mathematically, we can write;

$$\boxed{U = mgh}. \tag{6.9}$$

Example 1. Find the potential energy of a 100-kg block 10.0 m along a 30° inclined plane.

Answer:

Note, the block is 10 meters along an inclined plane, so we must find the vertical h first;

$$h = 10 \times \sin 30 \tag{6.10}$$

$\sin 30 = \frac{1}{2}$, therefore, $h = 5\ m$, then the potential energy of the block is;

$$U = mgh = 100 \times 9.8 \times 5 \tag{6.11}$$

$$\boxed{U = 490.0\ J} \tag{6.12}$$

6.1.3 Elastic Potential Energy

Elastic potential energy is the energy stored in elastic bodies such as a spring or a stretched rubber band. For example, this elastic energy can create a smooth ride in the suspension system of a car. We start from the definition of work as in equation 6.2 and substitute for force its functional dependence on position from the Hooke's law, i.e., $F = -kx$. We can therefore write,

$$dU = \int_{x_1}^{x_2} -kxdx \tag{6.13}$$

Integrating with respect to x;

$$U = \frac{1}{2}k(x_1^2 - x_2^2). \tag{6.14}$$

From equation 6.14 we can deduce that at any point in the stretched or compressed state of a spring we can write;

$$\boxed{U = \frac{1}{2}kx^2}. \tag{6.15}$$

It is noteworthy to emphasize that x is the amount of stretch or compression in the spring and not simply the position vector.

Example 2. Calculate the elastic potential energy stored in a spring with a spring constant $k = 800\ N/m$ and stretched by 10.0 cm.

Answer:

First we must convert cm to m.

$$x = 10.0/100 = 0.1\ m \tag{6.16}$$

Now using equation 6.15 we have;

$$U = \frac{1}{2}800(0.1)^2, \tag{6.17}$$

or;

$$\boxed{U = 4.0\ J} \tag{6.18}$$

6.2 Conservation of Energy

The conservation of energy or more accurately the conservation of mechanical energy refers to the old idea of material conservation where the total amount of matter is conserved. It is more appropriate to define this very important law as;

Definition: *The total mechanical energy of a system remains constant throughout the evolution of that system.*

This can be written mathematically as;

$$\boxed{T + U = C}. \tag{6.19}$$

It is important to make the observation that equation 6.19 is the definition of the conic section ellipse. Remembering our geometry, the ellipse is defined as the locus of all points on a plane where the sum of the distances from two fixed points called focus is a constant. In equation 6.19 the kinetic energy and potential energy can be thought of as the two distances from the two foci. As a matter of fact, as we shall see later in chapter 10 the planets are revolving around the Sun in an elliptical orbit with the Sun located in one of the two foci. This is the statement of Kepler's First Law of planetary motion.

We also should emphasize that the term mechanical energy refers to the work done by a class of forces called conservative forces. Conservative forces are functions of position only and can be expressed as the derivative of a potential with respect to the position vector, i.e.;

$$\boxed{F = -\frac{dU}{dr}}. \tag{6.20}$$

Note equation 6.20 can be rearranged in the integral form of 6.2.

Example 3. A spring gun has a spring constant $k = 20000.0\ N/m$. When the gun is cocked, the spring is compressed by 10.0 cm. If a steel ball with a

$m = 20.0\ g$ is then dropped into the barrel, calculate the velocity of the steel ball as it exits the barrel. Neglect all frictions.

Answer:

Since all friction is neglected then the forces are conservative and the elastic potential energy of the spring is equal to the kinetic energy of the steel ball. Hence, we have;

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2. \tag{6.21}$$

Plugging in the values with appropriate SI units, we can write;

$$\frac{1}{2}20000.0 \times 0.1^2 = \frac{1}{2}0.02v^2, \tag{6.22}$$

or;

$$\boxed{v = 100.0\ m/s}. \tag{6.23}$$

6.3 Problems

1. Calculate the work done when moving a $50 - kg$ crate a distance of $30\ m$ on a horizontal surface with a coefficient of dynamic friction $\mu_d = 0.15$.

2. A crane lifts a car $5\ m$. If the mass of the car is 1 *metric ton*, calculate the work done by the crane.

3. A man is pulling a $100 - kg$ sled at a constant speed up a snowy hill a distance of $50\ m$. If the angle of the hill is $30°$, find the amount of work done by the man.

4. What force is needed to move a cart a distance of $50\ m$ in a grocery store if the work done is $500\ J$? Assume the force is applied at an angle of $45°$.

5. A large forklift is moving a mobile home from one lot to another. If the applied force required to move the mobile home at an angle of $15°$ a distance of $50\ m$ is $3000\ N$, what is the work done?

6. A boy walking his dog is applying a force of 100 N on his dog at an angle of $30°$. How much work is done by the boy for a 0.5-km walk?

7. It takes $70\ J$ of work to push a desk 5 m across the floor, what force would be needed if applied at an angle of $20°$?

8. A strongman pulls an $80 - ton$ Boeing $737 - 800$ a distance of $50\ m$ in $1.7\ minutes$. If the force of resistance is 10% of the weight of the plane, calculate a) the work done by the strongman if he applies the force at an angle of $30°$, and b) calculate the power generated by the strongman.

9. Calculate the kinetic energy of a $700 - kg$ Volkswagen traveling at $60\ km/hr$. At what velocity would its kinetic energy increase by a factor of 3?

10. A uniform steel rod is pivoted at one end and is hanging from a ledge. It we deflect the rod upwards by $45°$, how much does its potential energy increase?

11. In old television sets, an electron hits the screen at a speed of $1.0 \times 10^4\ km/hr$. Calculate the kinetic energy of the electron. Look up the mass of the electron on the web.

12. A $1200 - g$ block is dropped from a height of $50\ cm$ onto a vertically mounted spring. Calculate the amount of compression of the spring if the spring constant is $1500\ N/m$.

13. A small bucket of water is tied to a $75\ cm$ long rope and is spun in a vertical plane. Calculate the minimum required speed at the top of the loop if no water spills out of the bucket.

14. A piece of ice with mass M is on the top of an ice-covered dome as shown in figure 6.1. If the ice starts to slide down the dome with negligible friction, at what angle θ would it separate from the dome?

15. The Atwood machine shown in figure 6.2 is released when the block M_2 is $2\ m$ above the floor. Find the velocity of the system when M_2 hits the ground.

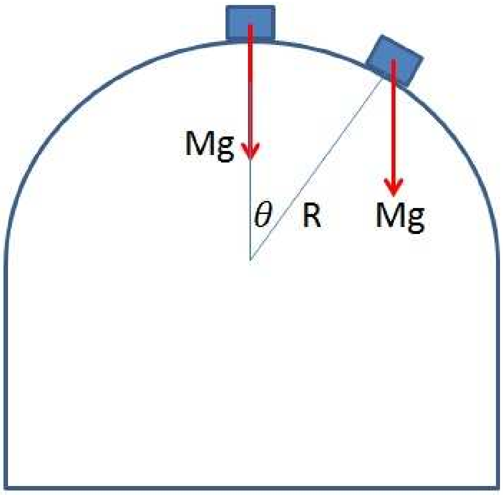

Figure 6.1: A block of ice on an ice covered dome just about to slide down.

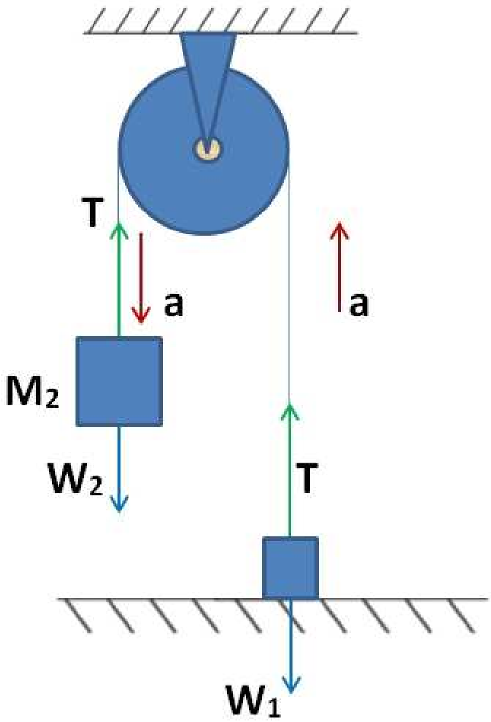

Figure 6.2: An Atwood machine problem demonstrating a conservation of energy problem.

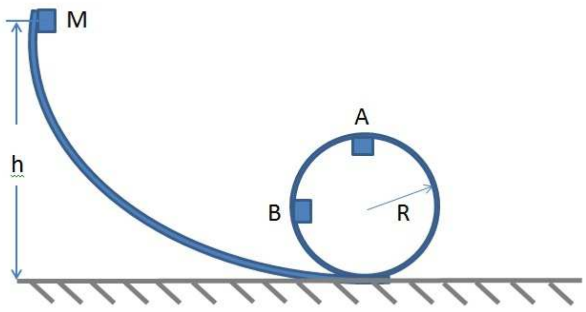

Figure 6.3: Loop-the-loop problem demonstrating use of conservation of energy.

16. Using *Conservation of Energy* in problem 15, calculate the acceleration of the system. Can you calculate the tension in the cord with this method? Quantify your answer.

17. An object with mass M is released from a height h in a loop-the-loop shown in figure 6.3. Use conservation of energy to find the ratio of $\frac{h}{R}$ if the object were to reach point A in the figure. What should this ratio be if it reaches point B? Neglect friction.

18. In the previous loop-the-loop problem find the centripetal acceleration at points A and B.

Chapter 7

Momentum, Impulse and Conservation of Momentum

In Chapter 6 we discussed work and energy and its various forms such as kinetic, potential and elastic energy. In this chapter we introduce and discuss a vector quantity called the momentum where we reformulate what we have learned so far using this important "motional" variable. Momentum is defined as the mass of the object times its velocity. We denote momentum with the Latin letter **p**. Mathematically, we can write;

$$\boxed{\mathbf{p} = m\mathbf{v}}. \tag{7.1}$$

Along with momentum, we also investigate the concept of *impulse* which is defined as the change in the momentum of an object.

Before we delve into a full mathematical treatment of momentum, impulse and conservation of momentum, we shall introduce the concept of *center of mass*. This topic has to be investigated since it has direct consequences for understanding scattering and decay which will be covered later in this chapter.

7.1 Center of Mass

The center of mass of an extended object, as opposed to a point particle assumed so far, is referred to as a point where the weighted sum of the distributed mass is zero. The object is balanced around the center of mass and its coordinates are the average of the weighted position coordinates of mass distribution.

Any extended irregular shaped object as shown in figure 7.1 can always be approximated as a collection of *lumped masses*.

With this in mind, we can obtain the coordinates of the center of mass of an object;

$$\sum_{i=1}^{n} m_i \mathbf{r}_i \approx M\mathbf{R} \tag{7.2}$$

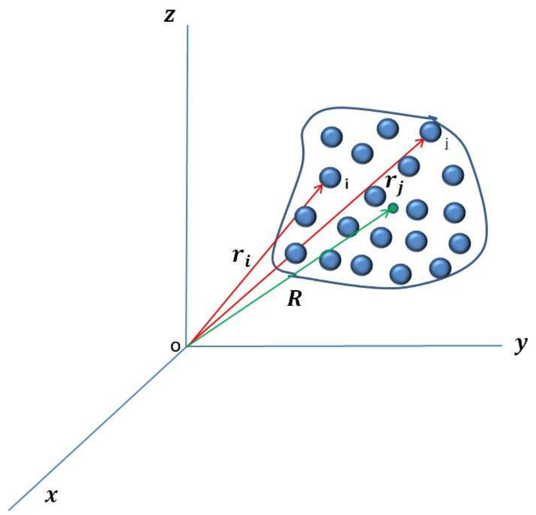

Figure 7.1: A lumped mass parameter depiction of an irregular shaped object.

$$\mathbf{R} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M}. \tag{7.3}$$

Recall, both $\mathbf{r_n}$ and $\mathbf{R}$ are vectors in three dimensions and can be expressed in the **ijk** notation.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \tag{7.4}$$

If the number of lumped masses approaches infinity, then for such a continuous body the summation will yield the exact value of the vector R and the sum will be replaced by an integral.

$$\mathbf{R} = \frac{1}{M} \int_V \rho(\mathbf{r})\mathbf{r} dV \tag{7.5}$$

In equation 7.5, $\rho(\mathbf{r})$ is the density of the object and V is the entire volume. Note, equation 7.5 can only be solved if $\rho(\mathbf{r})\mathbf{r}$ is integrable. This requirement is very difficult to achieve and that is why in most engineering applications a lumped-mass parameter approach using computers is utilized.

In modern airplanes, rockets and spacecrafts, due to the constant expenditure of fuel and movement of passengers, the center of mass continuously changes its location. An up-to-date knowledge of the location of the center of mass is required for navigation and control of the craft. The so-called fly by wire technology or computer aided control automatically takes care of these issues without human interference for course correction.

Example 1. Three masses $m_1 = 1.0\ kg$, $m_2 = 2.0\ kg$ and $m_3 = 3.0\ kg$ as shown in figure 7.2 are located in the x-y plane as shown with m_1 at $(0, 0)\ m$,

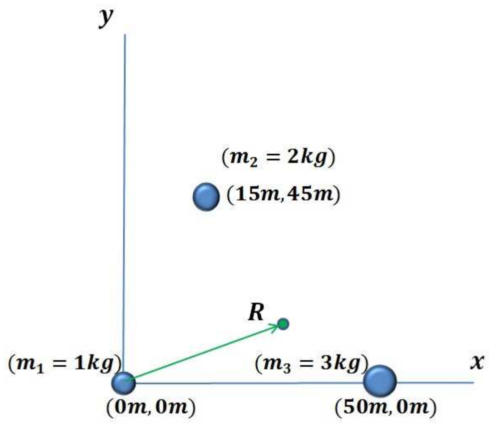

Figure 7.2: Three objects located at the vertices of a triangle.

m_2 at $(15, 45)$ m and m_3 at $(50, 0)$ m. Find the coordinate of the *center of mass.*

Answer:

We proceed according to equation 7.2;

$$X_{c.m.} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}, \tag{7.6}$$

and;

$$Y_{c.m.} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}. \tag{7.7}$$

Plugging in the values from figure 7.2 we get;

$$\boxed{X_{c.m.} = 30\ m}, \tag{7.8}$$

and;

$$\boxed{Y_{c.m.} = 15\ m}. \tag{7.9}$$

7.2 Newton's Second Law, revisited

Now we will reformulate the Newtonian mechanics, especially Newton's Second Law of motion in terms of momentum $\mathbf{p}$ rather than acceleration $\mathbf{a}$.

Recall Newton's Second Law is;

$$\mathbf{F} = m\mathbf{a}. \tag{7.10}$$

However,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}, \tag{7.11}$$

and the momentum is $\mathbf{p} = m\mathbf{v}$ and if mass is *not* a constant, then we can write;

$$\mathbf{F} = \frac{dm\mathbf{v}}{dt}, \tag{7.12}$$

or;

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}}. \tag{7.13}$$

The formulation presented in equation 7.13 is the actual statement of Newton's Second Law of motion as first published in Principa Mathematica in Latin in 1687.

We express this idea verbatim from Wikipedia.

Lex II: *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*

This was translated quite closely in Motte's 1729 translation as:

Law II: *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

According to modern ideas of how Newton used his terminology, this is understood, in modern terms, as an equivalent of:

Law II: *The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.*

Motte's 1729 translation of Newton's Latin continued with Newton's commentary on the Second Law of motion, reading:

"If a force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both."

As we mentioned previously, in practical engineering calculations, mass is not a constant and the rate of change of momentum which is mass times velocity is the force applied on the body.

7.3 Impulse

According to modern day interpretation of Newton's Second Law *Impulse* is the change of momentum of an object. We can define impulse shown with the Latin letter J by using equation 7.14;

$$\mathbf{dp} = \mathbf{F}dt. \tag{7.14}$$

Now let us integrate equation 7.14 with respect to time;

$$\mathbf{J} = \int_{t_1}^{t_2} \mathbf{F}dt, \tag{7.15}$$

however;

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{7.16}$$

Substituting for $\mathbf{F}$ from equation 7.16 in the integral equation 7.15, we obtain;

$$\mathbf{J} = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt}dt = \int_{p_1}^{p_2} d\mathbf{p}, \tag{7.17}$$

hence,

$$\boxed{\mathbf{J} = p_2 - p_1 = \Delta \mathbf{p}}. \tag{7.18}$$

The above equations tell us that the longer the object is in contact with another object the higher the impulse of **J** and therefore the higher the momentum transfer between the two objects.

7.4 Conservation of Momentum

The idea of the *Conservation of Momentum* comes from the old idea that the total amount of motion for a system is conserved. Therefore, we can define conservation of momentum among interacting objects or particles which are isolated from external forces as follows;

Definition: *In a isolated system, the total momentum is a constant in both direction and magnitude.*

This is a direct consequence of Newton's Third Law of action-reaction. Recall Newton's Third Law states that for every force, there is an opposite and equal reaction. This mathematically states the following;

$$\mathbf{F} = -\mathbf{F}. \tag{7.19}$$

However, using equation 7.16 we can write;

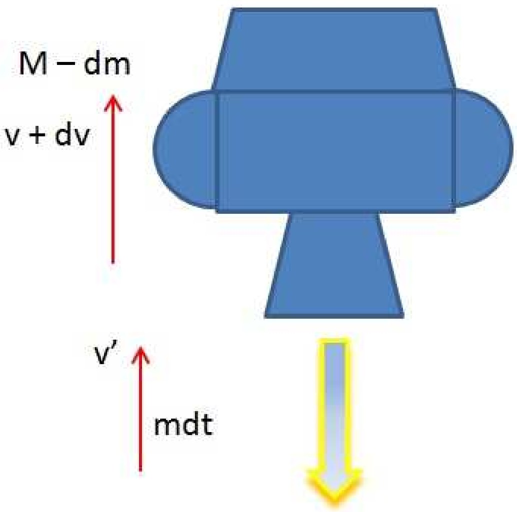

Figure 7.3: A depiction of the LEM taking off from the Moon.

$$\frac{d\mathbf{p}}{dt} = -\frac{d\mathbf{p}}{dt}, \tag{7.20}$$

or;

$$\frac{d}{dt}(\mathbf{p_1} + \mathbf{p_2}) = 0. \tag{7.21}$$

Equation 7.21 tells us mathematically that when the derivative of quantity is zero, then that quantity is a constant. We therefore can write;

$$\boxed{\mathbf{p_1} + \mathbf{p_2} = Constant}. \tag{7.22}$$

Equation 7.22 is the mathematical statement of the definition of conservation of momentum.

Example 2. The last Apollo Lunar Excursion Module (LEM) took off from the Moon on December 14, 1972 at 5:55 PM US Eastern Standard Time. If the mass of the LEM plus the fuel is M and the rate of the rocket exhaust is dm/dt find the mass M (rocket plus fuel) that took off from the moon; calculate the fuel to mass ratio for the system to attain lunar orbit and rendezvous with the Apollo command module. For Apollo 17, the exhaust velocity is 3050 m/s and burn time is 446.1 s and the velocity to attain lunar orbit was 1.687 km/s (http://www.braeunig.us/apollo/LM-ascent.htm). ($g_{Moon} = 1.6249\ m/s^2$)

Answer:

The total momentum of the system is Mv. In a short time interval dt, a mass dm of exhaust in ejected. Let v_r be the downward velocity of gas relative to the LEM. Then the velocity of the gas relative to the Moon is;

$$\mathbf{v}' = \mathbf{v} - \mathbf{v_r}a, \tag{7.23}$$

or, for the momentum we have;

$$dm\mathbf{v}' = dm(\mathbf{v} - \mathbf{v_r}). \tag{7.24}$$

As shown in figure 7.3. the LEM mass decreases to $(M - dm)$ and the velocity increases to $(v + dv)$, then;

$$p = (M - dm)(v + dv). \tag{7.25}$$

Therefore, the total momentum at $t + dt$ is;

$$p = (M - dm)(v + dv) + dm(v - v_r). \tag{7.26}$$

Since the LEM took off from the Moon, the absence of an atmosphere removes air resistance. The weight of the LEM multiplied by the dt provides the *Impulse* and it is the total change in the momentum.

$$-mg_{Moon}dt = (M - dm)(v + dv) + dm(v - v_r) - mv \tag{7.27}$$

Or;

$$-mg_{Moon}dt = Mdv - dmv_r + dmdv. \tag{7.28}$$

In equation 7.28 the product $dmdv$ is small and can be neglected, then;

$$-mg_{Moon}dt = Mdv - dmv_r, \tag{7.29}$$

dividing by dt;

$$-Mg_{Moon} = M\frac{dv}{dt} - v_r\frac{dm}{dt}, \tag{7.30}$$

or;

$$\boxed{M\frac{dv}{dt} = v_r\frac{dm}{dt} - Mg_{Moon}}. \tag{7.31}$$

The acceleration is obtained by dividing both sides of the equation 7.31 by M;

$$\frac{dv}{dt} = \frac{v_r}{M}\frac{dm}{dt} - g_{Moon}. \tag{7.32}$$

By separating the variables we can write;

$$dv = v_r \frac{dm}{dt} - g_{Moon} dt. \tag{7.33}$$

Integrating;

$$\int_{v_0}^{v_{orbit}} dv = \int_{M}^{m} v_r \frac{dm}{M} \int_{0}^{t} dt, \tag{7.34}$$

or;

$$\boxed{v_{orbit} = v_0 - g_{Moon} t + v_r \ln \frac{M}{m}}. \tag{7.35}$$

Because the LEM is starting from rest, then $v_0 = 0$. Plugging in the values for v_{orbit}, g_{Moon} and burn time t, we obtain the value for $\frac{M}{m}$.

$$1687m/s = -1.6249m/s^2 \times 446s + 3050m/s \ln \frac{M}{m}, \tag{7.36}$$

or;

$$\boxed{\frac{M}{m} = 2.2}. \tag{7.37}$$

7.5 Scattering

The topic of scattering is one of the most important and fundamental areas of Physics. At a microscopic level, the act of seeing an object is because of a scattering of photons from a given surface. We can see colors, shapes, texture and other attributes of an object by the process of scattering. There are two types of scattering, elastic and inelastic and we will discuss them in the following subsections.

7.5.1 Elastic Scattering

The idea of scattering provides a laboratory where both conservation of momentum and energy are applied. An elastic scattering is defined as one where there is no loss of kinetic energy. In reality, there is always loss of energy in any scattering process, but in some cases where the objects are rigid enough, then the process can be thought of as almost elastic scattering.

Mathematically we can show any elastic scattering process as;

$$T_i = T_f; and\ \mathbf{p_i} = \mathbf{p_f}. \tag{7.38}$$

We can show, as an example, how the two fundamental physics topics namely the conservation of energy and momentum can give us information regarding the evolution of interacting bodies.

Example 3. In figure 7.4 a particle of mass m_1 and initial velocity v_{1i} is making a glancing elastic collision with another particle initially at rest. After collision, the final velocity of the projectile is v_{1f} and its scattering angle is ϕ. Calculate the final velocity v_{2f} and its scattering angle θ.

Answer:

We have two unknowns in this problem, v_{2f} and θ. We therefore must write at least two equations for these two unknowns. Conservation of energy and momentum will provide the necessary equations.

First, conservation of energy;

$$T_i = T_f, \tag{7.39}$$

or;

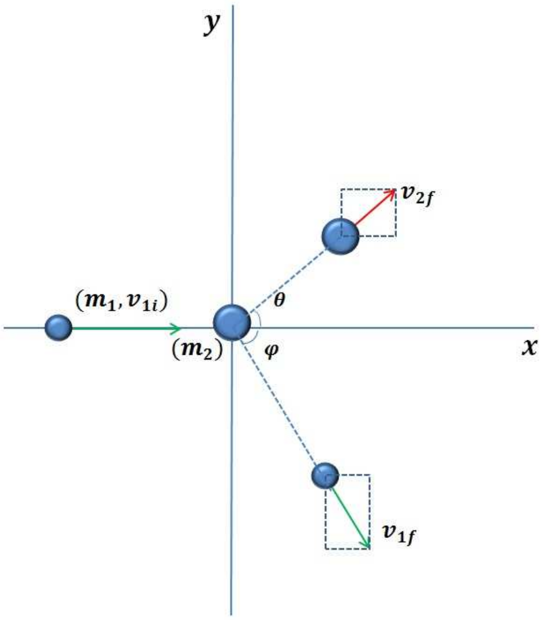

Figure 7.4: A particle with mass m_1 and velocity v_{1i} is making a glancing elastic collision with another particle initially at rest.

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2. \tag{7.40}$$

Canceling $\frac{1}{2}$ from both sides we can write;

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2. \tag{7.41}$$

The conservation of momentum is a vector equation and we must take the x and the y components to solve the vector equation.

$$\mathbf{p_i} = \mathbf{p_f}. \tag{7.42}$$

In the $x - direction$;

$$p_{1i} = p_{1f}\cos\phi + p_{2f}\cos\theta, \tag{7.43}$$

or;

$$m_1 v_{1i} = m_1 v_{1f}\cos\phi + m_2 v_{2f}\cos\theta. \tag{7.44}$$

In the $y - direction$;

$$m_1 v_{1f} \sin\phi = m_2 v_{2f} \sin\theta. \tag{7.45}$$

Known variables are m_1, m_2, v_{1i}, v_{1f} and ϕ. Dividing equation 7.45 by 7.44 we obtain;

$$\boxed{\tan\theta = \frac{m_1 v_{1f} \sin\phi}{m_1 v_{1i} - m_1 v_{1f} \cos\phi}}. \tag{7.46}$$

From equation 7.41 we find the final velocity of the target mass as an ejectile.

$$m_2 v_{2f}^2 = m_1 v_{1f}^2 - m_1 v_{1i}^2, \tag{7.47}$$

or;

$$\boxed{m_2 v_{2f}^2 = \sqrt{m_1 v_{1f}^2 - m_1 v_{1i}^2}}. \tag{7.48}$$

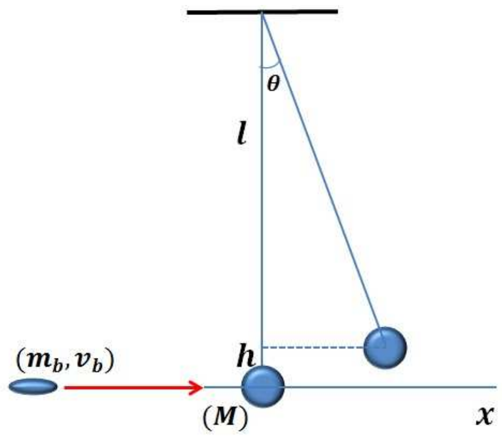

Figure 7.5: *Ballistic Pendulum* is a device for measuring the velocity of a projectile or a bullet.

7.5.2 Inelastic Scattering

Inelastic scattering is one in which the kinetic energy is not totally conserved and changes to some other type of energy such as heat. Unlike energy, momentum is conserved in inelastic scattering. Therefore, the "motional" conservation law is a more fundamental conservation law than "material" conservation law. The best way to illustrate the concept of the inelastic scattering is through the example of the *Ballistic Pendulum*.

Example 4. A Blackwood *Ballistic Pendulum* is a simple pendulum of length l with an impact absorbing bob as shown in figure 7.5. The projectile or the bullet with mass m_b and velocity v_b is fired horizontally at the bob with the mass M and comes to rest inside the bob and raises it to height h above the horizon. The only unknown variable is the v_b and we will calculate it from the laws of conservation momentum and energy.

Answer:

From conservation momentum we can write;

$$m_b v_b = (m_b + M)V. \tag{7.49}$$

There are two unknowns in equation 7.49, v_b and V. We can calculate V by using conservation of energy after the collision.

$$\frac{1}{2}(m_b + M)V^2 = (m_b + M)gh. \tag{7.50}$$

We can express h in terms of l and $\cos\theta$.

$$h = l(1 - \cos\theta). \tag{7.51}$$

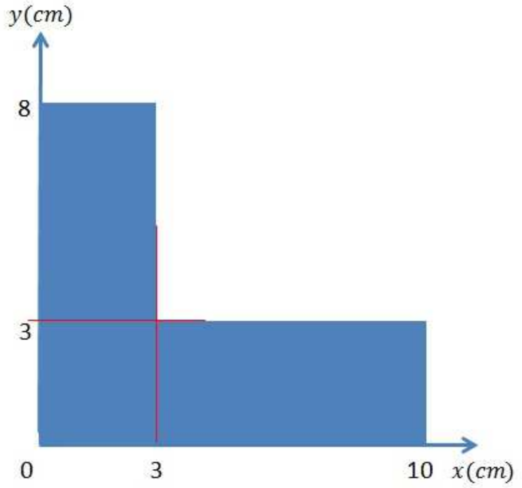

Figure 7.6: *A L-shaped thin metallic object with uniform surface density.*

Therefore, from equations 7.50 and 7.51;

$$V = \sqrt{2l(1 - \cos\theta)}. \tag{7.52}$$

Substituting V in 7.50 we have;

$$m_b v_b = (m_b + M)\sqrt{2l(1 - \cos\theta},\tag{7.53}$$

or;

$$\boxed{v_b = \frac{(m_b + M)\sqrt{2l(1 - \cos\theta)}}{m_b}}. \tag{7.54}$$

7.6 Problems

1. Four masses $m_1 = 2.0\ kg$, $m_2 = 1.5\ kg$, $m_3 = 1.0\ kg$ and $m_4 = 2.5\ kg$ are located in the $x - y$ plane with m_1 at $(0, 1)\ m$, m_2 at $(10, 40)\ m$, m_3 at $(30, 2)\ m$ and m_4 at $(-25, -10)\ m$. Find the coordinate of the center of mass.
2. A thin L-shaped metal object, figure 7.6, has the dimensions shown and has a uniform surface density of $\sigma\ g/cm^2$. Calculate the coordinates of the center of mass.
3. A force is applied for 1.0 μs on an object, find its momentum.
4. Write down the equation for the momentum of an object with uniform acceleration and then find the force acting on the object.

5. Calculate the momentum of a $0.5 - ton$ race car moving at $150.0\ km/hr$. What is the speed of a $5.0 - ton$ truck with the same momentum?

6. Calculate the impulse of a force of $100.0\ N$ acting on a ball for 0.1 second.

7. A baseball is pitched at a velocity of $30.0\ m/s$. If the mass of the baseball is $150.0\ g$, find its momentum. If the batter hits the baseball head-on and the velocity in the opposite direction is $45.0\ m/s$, calculate the total change in the momentum of the ball.

8. In problem 7, if the time of the contact between the ball and the bat is $0.05\ s$, find the force that the baseball experiences.

9. The kinetic energy of an object is $50.0\ J$ and its momentum is $15.0\ kg - m/s$. Calculate the mass of the object.

10. In order to latch on a second car, a $10 - ton$ railroad car is moving with a speed of $4.0\ km/hr$ and collides with a second identical car resting on the rails. Find the speed of the two car system. Find the kinetic energy after the latching and compare it with the kinetic energy of the initially moving car.

11. In the example of the ballistic pendulum in this chapter, assume that a bullet with a velocity of $300.0\ m/s$ is not stopped in the block and that it comes out the other side. If the mass of the bullet is $5.0\ g$ and the mass of the wooden block is $1.0\ kg$ and it is attached to a string $1.5\ m$ long, and the center of mass of the block rises by $6.0\ cm$, calculate the velocity of the bullet as it exits the block.

12. A block of mass m is held in place on the top of a smooth inclined plane at a height h. If the inclined plane has a mass M and is resting on a frictionless horizontal floor and we release the block what is the velocity of the inclined plane when the block reached the floor?

13. A boat with a mass M with two people on board with masses m_1 and m_2 is moving with a velocity V_0. One person decides to take a swim and walks in the opposite direction in order to dive into the water. What is the speed of the boat and its remaining occupant after the first person has jumped out if their relative speed is v_r?

14. An α-decay is when a radioactive element emits an α particle which is the nucleus of a He atom. ^{238}U decays by α emission as:

$$^{238}U \rightarrow ^{234} Th + \alpha. \tag{7.55}$$

After looking up the masses of ^{238}U, ^{234}Th and α on the web, you will find out how much nuclear binding energy was available and was converted into kinetic energies of ^{234}Th and the α. Calculate the momentum and the kinetic energy of each decay particle.

15. A projectile of mass m makes a glancing collision with an equal mass target initially at rest. Prove that the angle between the the two scattered objects is $90°$.

16. Show that in an elastic collision between a projectile of mass M and a target at rest with mass m $(M > m)$, the maximum scattering angle of the projectile is $\cos\theta_{max} = \sqrt{1 - m^2/M^2}$.

17. Inside stars, including our Sun, due to the high temperature of the inner core the so-called *pp* cycle causes the hydrogen nuclei to fuse into deuterons and generate heat. If the speed attained by a deuteron is $10^7\ m/s$ due to the enormous heat and it collides with another deuteron with the same velocity and making an angle of $60°$ with the direction of the first d and they fuse to produce a He nucleus, find the kinetic energy of the He nucleus. Look up masses for d and He on the web.

18. A $1000-g$ wooden block is resting on a surface with a coefficient of kinetic friction $\mu_k = 0.15$. If we fire a $5.0-g$ bullet horizontally with a velocity of $250\ m/s$ at the block and the bullet comes to rest in the block, calculate how far the block and the bullet assembly would travel on the surface before stopping.

19. It is proposed that the Moon was formed due to a giant collision between a Mars-sized planetoid and the Earth. If the combined mass of the two bodies just after collision is the combined mass of today's Earth and the Moon, find the mass of the proto-Earth (look up masses of the Earth, the Moon and Mars on the web). If the present gravity of the Sun is $5.93 \times 10^{-3}\ m/s^2$ calculate the orbital velocity of the Earth.

20. In problem 19, if the planetoid was moving in the same direction as the proto-Earth with a velocity of $35\ km/s$, find the velocity of the proto-Earth. Calculate the distance of the proto-Earth from the Sun.

Chapter 8

Rotation

In Chapters 3 through 7 we discussed classical mechanical quantities such as displacement, velocity, acceleration, force, work, energy and momentum in rectilinear formalism where the coordinates were assumed to be x, y, and z. We briefly discussed angular displacement and velocity in chapter 4 where we introduced uniform circular motion. In this chapter we introduce classical mechanics in terms of angular variables.

The algebra, as we will see, is analogous to the linear coordinates and we will see, for example, how Newton's Second Law manifests itself in rotational dynamics.

It is also interesting to note that rotational motion is closer to reality. For example, the Earth rotates around its axis, the planets revolve around the Sun, the Sun revolves around the galaxy and so on. Furthermore, rotation also reveals the continuous repetition aspect and hence the periodic nature of the universe. Four seasons are repeated continuously and if there are no dates one spring looks just like any other one!

8.1 Angular Displacement

Angular displacement denoted by the Greek letter θ is expressed in terms of *radian*, and recalling our basic geometry it has the following relation to linear displacement or the "arc" **s** through the radius of rotation **R**;

$$\boxed{\mathbf{s} = R\theta.} \tag{8.1}$$

8.2 Angular Velocity

Angular velocity denoted by the Greek letter ω has the units of radian per second, and recalling chapter 3 analogous to the linear velocity, it is the derivative of the angular displacement θ with respect to time. In other words, angular velocity is a measure of how fast a given angle increases or decreases.

$$\boxed{\omega = \frac{d\theta}{dt}}. \tag{8.2}$$

The relation between angular velocity ω and the linear velocity v is;

$$\boxed{\mathbf{v} = R\omega}. \tag{8.3}$$

8.3 Angular Acceleration

Angular acceleration denoted by the Greek letter α is a measure of how rapidly the angular velocity increases with respect to time. By this definition the unit of angular acceleration would be rad/s^2. Mathematically, angular acceleration is the derivative of angular velocity with respect to time or the second derivative of the angle θ with respect to time. We therefore can write;

$$\boxed{\alpha = \frac{d\omega}{dt}}, \tag{8.4}$$

or;

$$\boxed{\alpha = \frac{d^2\theta}{dt^2}}. \tag{8.5}$$

The relation between angular acceleration ω and the linear acceleration a is;

$$\boxed{\mathbf{a} = R\alpha}. \tag{8.6}$$

For constant angular acceleration, i.e., $\alpha = C$, we can write as in chapter 3 for linear acceleration,

$$\alpha = C \tag{8.7}$$

Integrating equation 8.7 with respect to time we obtain the equation for angular velocity ω;

$$\omega = \int \alpha dt, \tag{8.8}$$

or;

$$\boxed{\omega = \alpha t + \omega_{\mathbf{0}}}. \tag{8.9}$$

Integrating equation 8.9 with respect to time we obtain the equation for angular displacement θ;

$$\boxed{\theta = \frac{1}{2}\alpha t^2 + \omega_{\mathbf{0}} t + \theta_{\mathbf{0}}}. \tag{8.10}$$

Note these angular kinematic equations are the analogs of linear kinematics discussed in chapter 3.

8.4 Torque

The analog of force in rotation is *Torque* and it is defined as;

$$\tau = \mathbf{r} \times \mathbf{F} \tag{8.11}$$

The cross product of two vectors τ is called an axial vector as was described in chapter 2. Note, the direction of the torque changes if the direction of the force changes.

The magnitude of this cross product vector τ is;

$$\tau = rF \sin\theta. \tag{8.12}$$

Note, θ is the angle between $\mathbf{r}$ and $\mathbf{F}$ and it yields the maximum torque when the angle is $90°$.

A good example of this is the right-handed screw. Imagine a bolt being tightened by a wrench with a long arm. The force exerted on the handle is not enough to do the job. We should also apply the force as far as possible from the center of rotation of the bolt. This is an intuitive reaction on our part to choose the longest possible lever arm and equation 8.11 shows this fact mathematically. For a given force, a longer lever arm, r, creates a larger torque.

8.5 Angular Momentum

The analog of linear momentum, discussed in chapter 7, is angular momentum denoted by the Latin letter $\mathbf{L}$. The angular momentum is expressed in terms of linear momentum and the radius of rotation r as;

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{8.13}$$

The magnitude of $\mathbf{L}$ is;

$$L = rp \sin\theta. \tag{8.14}$$

Again the maximum angular momentum corresponds to a $90°$ angle between $\mathbf{r}$ and $\mathbf{p}$.

8.6 Newton's Second Law in rotational motion

At this juncture, we will proceed to derive, using equations 8.11 and 8.13, Newton's Second Law of rotation. Keeping in mind the relations between linear and angular velocity from equation 8.3, we can write;

$$\tau = \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{8.15}$$

Differentiating equation 8.13 with respect to time we get;

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt}, \tag{8.16}$$

and we also know that;

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{8.17}$$

Differentiating a cross product is just like ordinary products but we must make sure the order of multiplication is respected, as it was described in chapter 2.

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{8.18}$$

Substituting for $\frac{d\mathbf{r}}{dt}$ from equation 8.17 and recalling $\mathbf{p} = m\mathbf{v}$, we have;

$$\frac{d\mathbf{L}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{8.19}$$

Note, the first term in 8.19 is the cross product of two parallel vectors which is zero. Hence, we can write;

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{8.20}$$

The right hand side of equation 8.20 is simply the torque; we therefore have;

$$\boxed{\tau = \frac{d\mathbf{L}}{dt}}. \tag{8.21}$$

A careful inspection of equation 8.21 reveals that it is the analog of Newton's Second Law in linear motion and therefore it is Newton's Second Law in rotational motion.

8.7 Angular Momentum, Rotational Kinetic Energy and the Moment of Inertia

Analogous to mass which we defined in chapter 4, as the ability of a body to resist acceleration, *Moment of Inertia* is the ability of an object to resist angular acceleration. This definition is crucial to identify $\tau = I\alpha$ as Newton's Second Law in a rotational frame. Although moment of inertia, denoted by the Latin letter I, is the analog of mass in rotation, it is a lot more complicated than mass and one needs to include geometry in order to calculate the moment of inertia, I. As we shall see, I actually gives us information regarding the distribution of the mass in a given body or in other words the shape or the geometry of the object.

Let us start with the angular momentum for a system of particles. We can write a vector equation;

$$\mathbf{L} = \mathbf{l_1} + \mathbf{l_2} + \mathbf{l_3} + ... + \mathbf{l_n}, \tag{8.22}$$

or;

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{l_i}. \tag{8.23}$$

Now we can write the angular momentum of each individual point mass in a given extended rigid body as;

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r_i} \times \mathbf{p_i}, \tag{8.24}$$

or;

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{r_i} \times \mathbf{v_i}. \tag{8.25}$$

Substituting $r\omega$ for v, we have;

$$\mathbf{L} = \sum_{i=1}^{n} m_i r_i^2 \omega. \tag{8.26}$$

Note the quantity $\sum_{i=1}^{n} m_i r_i^2$ is called the *Moment of Inertia* or I. As one can see it provides information regarding the distribution of mass throughout the body. Therefore, we can write;

$$\boxed{\mathbf{L} = I\omega}. \tag{8.27}$$

Equation 8.27 is the rotational analog of linear momentum $\mathbf{p} = m\mathbf{v}$.

Newton's Second Law can also be derived from equation 8.27 by using equation 8.21.

$$\tau = \frac{Id\omega}{dt} \tag{8.28}$$

Recall $\alpha = \frac{d\omega}{dt}$, hence we can write;

$$\boxed{\tau = I\alpha}. \tag{8.29}$$

This is the rotational analog of Newton's Second Law, $\mathbf{F} = m\mathbf{a}$.

Rotational kinetic energy can be derived from the linear kinetic energy by substituting $r\omega$ for v. We therefore can write;

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2, \tag{8.30}$$

or;

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2 \omega. \tag{8.31}$$

And finally we can write;

$$.\boxed{T = \frac{1}{2}I\omega^2} \tag{8.32}$$

8.8 Calculation of Moment of Inertia

The moment of inertia can be measured experimentally by applying a known torque and measuring the angular acceleration α. To calculate the moment of inertia for a system composed of discrete masses, we can use the defining equation $\sum_{i=1}^{n} m_i r_i^2$.

Example 1. A rigid body can be approximated by three masses as shown in figure 8.1. Calculate the moment of inertia about an axis perpendicular to the center of the equilateral triangle.

Answer:

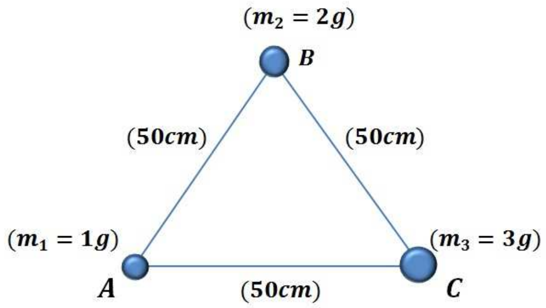

Figure 8.1: A lumped mass parameter approximation of a triangular object.

The center of a triangle is the point at which its medians intersect. Recalling our basic plane geometry, this point is $\frac{2}{3}$ of the way down on the median from a vertex.

Now from trigonometry we have to calculate the distance of one of the vertices from the center of the triangle. If we call the center O, we then can write;

$$AO = \frac{2}{3} AB \sin\theta \tag{8.33}$$

$\theta = 60°$ for an equilateral triangle and plugging in values from figure 8.1 we obtain;

$$AO = \frac{2}{3}(50)(\frac{\sqrt{3}}{2}) = 28.87\ cm \tag{8.34}$$

Because it is an equilateral triangle, all vertices are the same distance from the center. Now we can calculate the moment of inertia in the following manner;

$$I = \sum_{i=1}^{n} m_i r_i^2. \tag{8.35}$$

There are three masses, therefore, $n = 3$;

$$I = (1)(28.87)^2 + (2)(28.87)^2 + (3)(28.87)^2, \tag{8.36}$$

or;

$$\boxed{I = 173.22\ g.cm^2}. \tag{8.37}$$

For a continuous body, equation 8.35 changes into an integral form;

$$I = \int r^2 dm. \tag{8.38}$$

Or we can also write equation 8.38 in terms of volume V and the density ρ;

$$\boxed{I = \int r^2 \rho dV} \tag{8.39}$$

Example 2. Calculate the moment of inertia of a rod ($r \ll l$) shown in figure 8.2. The rod is solid with a density of ρ and it revolves around an axis perpendicular to the side and passing through its center O.

Answer:

We use equation 8.39 and keep in mind that we have to integrate over the length of the cylinder from $-\frac{l}{2}$ to $+\frac{l}{2}$. Therefore;

$$dV = \pi r^2 dx. \tag{8.40}$$

Note, dx is an element of length as shown in figure 8.2. Since the density ρ is a constant, we can write;

$$I = \pi \rho r^2 \int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx. \tag{8.41}$$

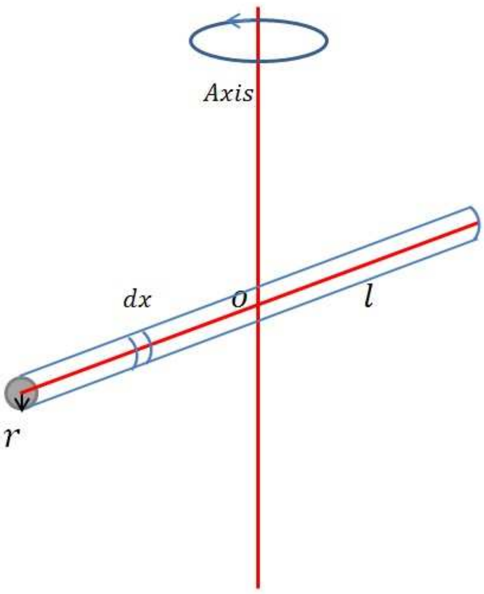

Figure 8.2: A rod rotating around an axis going through its center.

Integrating and plugging in the limits for x we obtain;

$$I = \frac{1}{3}\pi\rho r^2(\frac{l^3}{4}). \tag{8.42}$$

Note, the total mass of the rod M is $\pi\rho r^2 l$, therefore;

$$\boxed{I = \frac{Ml^2}{12}} \tag{8.43}$$

From this example, we see that the limits of integration play a crucial role in determining the moment of inertia. For example if we were to calculate the moment of inertia about an axis at one of the two ends of a rod, then the limits would change to $0 \leq x \leq l$ and the $I = \frac{Ml^2}{3}$ in that case.

8.8.1 Parallel Axis Theorem

The *parallel axis theorem*, or *Huygens-Steiner theorem* states that moment of inertia about any given axis is the sum of the moment of inertia of the body about the axis passing through its center of mass and Md^2, where d is the distance between the $c.m.$ and the axis of rotation. Therefore, we want to prove that;

$$I = I_{c.m.} + Md^2 \tag{8.44}$$

Proof:

Without the loss of generality, we consider the rod in the previous example. Using equation 8.38 and noting we must put in $x - d$ instead of just x;

$$I = \int (x-d)^2 dm \tag{8.45}$$

Squaring and integrating we obtain;

$$I = \int x^2 dm + d^2 \int dm - 2m \int x dx \tag{8.46}$$

Inspection of equation 8.46 reveals that the first term on the right hand side of this equation is simply $I_{c.m.}$ and the second term is Md^2. The last term is zero because the origin of the coordinate system is at the center of mass of the rod.

8.9 Conservation of Angular Momentum

Analogous to the conservation of momentum discussed in chapter 7, total angular momentum of a system is always conserved.

Conservation of angular momentum states that *if the sum of external torques on an object is zero then the angular momentum is conserved and the object is at rest or in a constant angular motion.*

Mathematically we can write;

$$\sum \tau = \frac{d\mathbf{L}}{dt} = 0, \tag{8.47}$$

or;

$$\mathbf{L} = C. \tag{8.48}$$

Because $\mathbf{L} = I\omega$, equation 8.48 tell us that any change in I requires a change in the angular velocity. Both $\mathbf{L}$ and ω are vectors, therefore, changes in one will change not only the magnitude but also the direction of the other one.

Mathematically we can write;

$$I_1\omega_{\mathbf{1}} = I_2\omega_{\mathbf{2}} \tag{8.49}$$

A good example of conservation of angular momentum is the spin of the figure skater. The slow twirl starts with arms stretched out. As the skater slowly raises arms overhead, she/he changes her/his geometry and mass distribution and therefore I. The skater goes from a large I to a small I and thereby increases her/his angular velocity and spins much faster.

The effect of conservation of momentum could become catastrophic on flight platforms such as helicopters. A helicopter has a main rotor and a tail rotor. Therefore the entire system has a total angular momentum. But if the tail rotor fails, the copter can go into a tail spin and crash.

8.10 Problems

1. The Moon completes an orbit around the Earth is 27.3 *days*. Find its angular displacement is 30 *hours*.

2. In problem 1, find the orbital velocity of the Moon. If the distance of the Moon from the Earth is 380000 *km*, find the centripetal acceleration of the Moon.

3. If the distance of the Earth to the Sun is 1.5×10^8 *km* find the orbital velocity of the Earth around the Sun.

4. Find the centripetal acceleration of the Earth and compare it to that of the Moon in problem 2.

5. If the mass of Mars is 0.107 of that of the Earth and its distance is 2.28×10^8 *km*, find the orbital angular momentum of Mars.

6. A circular saw completes 330 *rads* in 4.0 *s*. The angular velocity of the wheel after 4.0 *s* is 110.0 *rad/s*, calculate the angular acceleration of the wheel.

7. In problem 6, if the radius of the saw is 10.0 *cm* find the angular momentum of a particle at the tip of the teeth.

8. Calculate the moment of inertia of a sphere.

9. Calculate the angular momentum of the Sun and compare it to the orbital angular momentum of Jupiter. Look up all required parameters on the web.

10. Calculate the moment of inertia of a cylinder rotating around its central axis.

11. A mass M attached to a cord which is wrapped around a thin disc with mass m and radius R is shown in figure 8.3. Find the acceleration of the block and the tension in the cord.

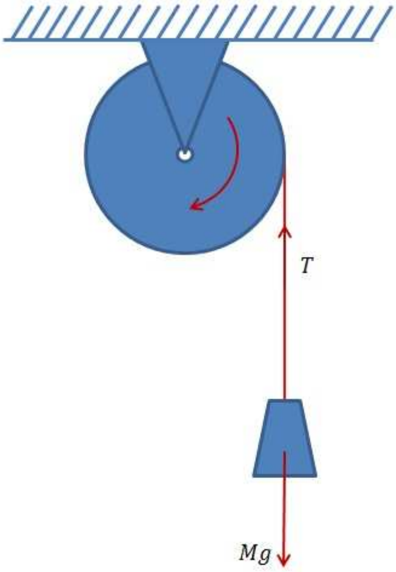

Figure 8.3: A hanging mass attached to a cord wrapped around a disc.

12. Repeat problem 11 if the block is sliding down an inclined plane with an inclination angle θ.

13. Calculate the tension in the cord and the acceleration of an Atwood machine if the pulley is a thin disc with mass m and radius R.

14. A wheel with mass M and radius R, shown in figure 8.4, is about to climb over a step of height h. What minimum force $\mathbf{F}$ is required to achieve this goal. At what height h does this task become impossible?

15. Calculate the rotational kinetic energy of the Earth and the Sun. Look up all required parameters on the web.

16. A thin hoop and a sphere both with mass m and radius r start rolling from the top of an inclined plane at a height h. Find the the velocity of the center of mass for each one at the bottom of the incline.

17. A high performance engine produces 400 hp when rotating at a speed of 1800 revolutions/minute. Calculate the torque it delivers. ($1\ hp = 745.699872\ W$).

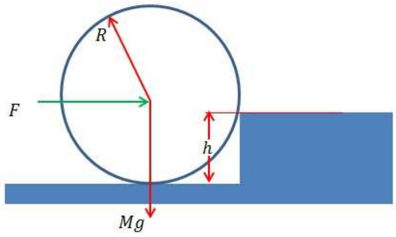

Figure 8.4: A wheel about to climb over a step.

18. The Moon's orbital period is 27.3 days. If the average distance of the Earth-Moon system is 380000 km, a) find the orbital angular momentum of the Moon. b) Calculate the orbital angular momentum of the Moon around the Sun and compare with that of part a. c) Calculate the angular momentum of the Earth as it spins around its axis and compare it with the result of part a.

19. If the ice on the two poles melts, the Earth would be closer to a sphere than an ellipsoid. If the moment of inertia of an ellipsoid of revolution is $I = \frac{1}{5}M(a^2 + b^2)$, where a and b are the equatorial and the polar radii of the Earth, calculate how long a day would be on a spherical Earth.

20. The angular momentum of a system with a moment of inertia $I = ml^2$ has a time dependence of $L(t) = ml^2(5t+3)$. What is the angular acceleration of the system? Calculate the torque the system generates, if $m = 5.0\ kg$ and $l = 50.0\ cm$.

21. In the problem 20, calculate the total kinetic energy of the system.

22. Two ice dancers with moments of inertia I_1 and I_2 are rotating on a fixed axis normal to the ice with angular velocities ω_1 and ω_2, respectively. If their angular momenta are L_1 and L_2 and they join hands what is their angular momentum after joining. Find their kinetic energy before and after joining.

Chapter 9

Statics and Elasticity

In Chapters 3 through 8 we discussed classical mechanical quantities such as displacement, velocity, acceleration, force, work and energy, and momentum in both translational and rotational motion. In this chapter, we shall investigate the application of Newton's First Law for both linear and rotational dynamics to analyze the mechanics of a body in static equilibrium. This is a very important topic in *Continuum Mechanics* where calculation of forces and torques on elastic bodies are required. For example, it is crucial to know the forces acting on a column supporting a bridge in order to design the column to be able to withstand the forces.

9.1 Static Equilibrium

When we look at a high-rise or a bridge or a ship, we note that these structures are put together tightly and parts in the structure are not moving. This observation is one of the design criteria for buildings and other structures. For example, when you are sitting on your seat on a airplane, you do not want to have a wobbly seat or a seat that is about to come off the floor of the plane. Even if the seat does not separate from the floor, it is still physically and psychologically uncomfortable to sit on a loose seat. The mathematical requirement for static equilibrium is due to Newton's First Law;

$$\sum \mathbf{F} = 0, \tag{9.1}$$

and;

$$\sum \tau = 0. \tag{9.2}$$

These two equations insure that the body is either at rest or at constant velocity/constant angular velocity. For structures, we do not want the body to move or rotate. When you are driving on a bridge, the bridge had better be at rest, i.e., static equilibrium or else you would be in trouble.

In Civil and Structural Engineering the term *Moment* is used instead of torque and is denoted with the capitol Latin letter M. Also the notion for the

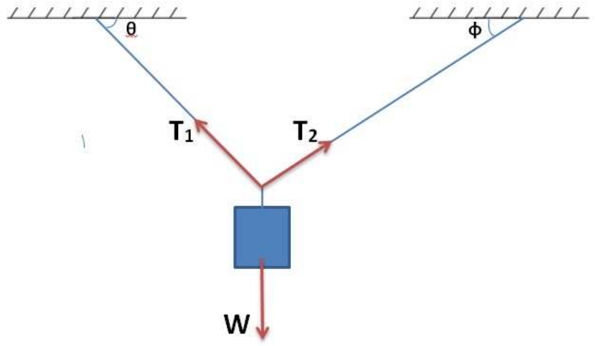

Figure 9.1: An object with a hanging mass in static equilibrium.

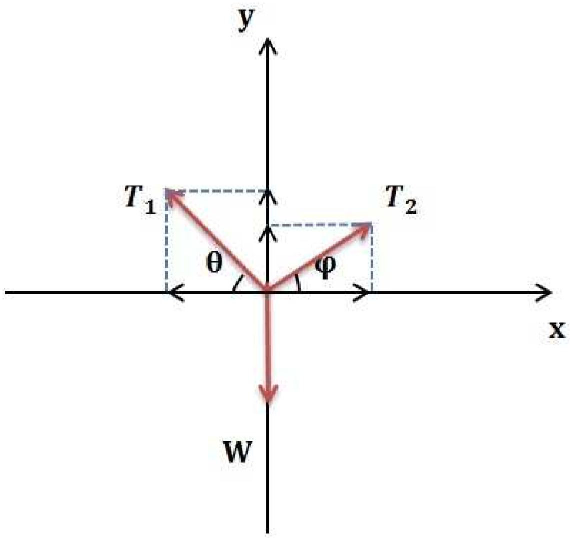

Figure 9.2: The free body diagram of the figure 9.1.

direction of the *Moment* vector is usually expressed in terms of clockwise or counter clockwise directions. The angle between the moment arm r and the force F is $90°$ or it is made to be $90°$ by resolving the force into its components along r and perpendicular to r.

Example 1. Find the tension in the cable supporting the weight of hanging mass M as shown in figure 9.1

Answer:

This example provides a venue by which we can introduce the idea of *Free Body Diagram*. It is a rough sketch used by engineers to simplify a system in terms of forces acting upon it. Figure 9.2 shows the free body diagram for figure 9.1.

We have two unknowns T_1 and T_2, therefore we need two equations. These two equations are provided by taking the components of T_1 and T_2 along the x and the y directions.

$$\sum F_x = 0, \tag{9.3}$$

and;

$$\sum F_y = 0. \tag{9.4}$$

In the $x - direction$ we can write;

$$\sum F_x = T_1 \cos\theta - T_2 \cos\phi = 0, \tag{9.5}$$

and; in the $y - direction$ we can write;

$$\sum F_y = T_1 \sin\theta + T_2 \sin\phi - W = 0. \tag{9.6}$$

Note, $W = Mg$ is the weight of the hanging mass. Now that we have our two equations for T_1 and T_2, we can proceed with the algebra and evaluate the unknowns.

$$T_1 = \frac{T_2 \cos\phi}{\cos\theta} \tag{9.7}$$

Substituting for T_1 from equation 9.7 in equation 9.6 we obtain $T2$;

$$\boxed{T_2 = \frac{W}{\cos\phi\tan\theta + \sin\phi}}, \tag{9.8}$$

and T_1;

$$T_1 = \frac{W\cos^2\phi}{\cos\phi\sin\theta + \sin\phi\cos\theta}. \tag{9.9}$$

However, $\cos\phi\sin\theta + \sin\phi\cos\theta = \sin(\theta+\phi)$ and hence we can write;

$$\boxed{T_1 = \frac{W\cos^2\phi}{\sin(\theta+\phi)}}. \tag{9.10}$$

This example could be a model for a suspension bridge. For engineering design purposes the knowledge of the amount of tension is crucial for the choice of strong enough cables in constructing the bridge.

9.2 Center of Gravity

In chapter 7 we discussed the notion of *Center of Mass* and how we can replace the motion of an extended body by the motion of its *Center of Mass*. In this section we introduce the notion of *Center of Gravity*.

Definition: *Center of Gravity of an object is a single point in a body where the net torque due to the force of gravity, i.e., weight is zero.*

Near the surface of a large mass body such as the Earth or the Moon where the acceleration of gravity is uniform and parallel throughout the object, *Center of Mass* and *Center of Gravity* are the same.

$$\sum \tau = \int_V (\mathbf{r} - \mathbf{R}) \times \mathbf{w}(\mathbf{r}) = \left(\int_V \rho(\mathbf{r})(\mathbf{r} - \mathbf{R}) dV \right)(-g). \tag{9.11}$$

From equation 9.11 we see that if $\sum \tau = 0$, and g is not a function of r, then equation 9.11 reduces to the definition of *Center of Mass* defined in chapter 7.

Now we will look at another case of static analysis which cannot simply be solved with forces alone. We must also analyze the torque or moment to determine all forces exerted on the system.

Example 2. Shown in figure 9.3 is a beam of length L which is supported at both ends. The beam has a weight of W_B and is carrying a load W resting at a

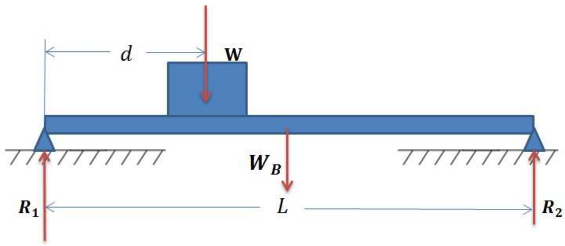

Figure 9.3: A beam of weight W_B supporting a weight W.

distance d from the left support as shown. Calculate the reactions at two ends R_1 and R_2.

Answer:

Figure 9.3 can be an approximate model for a bridge. The object can be a car and for design purposes, we need to find out the reactions of the supports R_1 and R_2. However, we see we have two unknowns and now we must require both the sum of the forces and the sum of the moments or the torques around a given point on the beam.

Referring to figure 9.3, we can write the sum of the forces in the $y-direction$ and we note that there are no forces in the $x - direction$ in this particular example.

$$\sum F = 0. \tag{9.12}$$

Substituting for F, we have;

$$R_1 + R_2 - W - W_B = 0. \tag{9.13}$$

The second equation comes from summing the moments around support 1; we assume all counter clockwise moments to be positive.

$$\sum M_1 = 0, \tag{9.14}$$

or;

$$(R_2)(L) - (W)(d) - (W_B)(L/2) = 0. \tag{9.15}$$

Note the weight of the bridge is acting on its center of gravity midway in the beam. From equation 9.15, we find R_2;

$$\boxed{R_2 = \frac{2Wd - W_B L}{2L}} \tag{9.16}$$

Substituting for R_2 in equation 9.13, we can write;

$$\boxed{R_1 = W + W_B - \frac{2Wd - W_B L}{2L}} \tag{9.17}$$

9.3 Elasticity

In the previous section and throughout the text so far we assumed that the body is rigid and there is no internal motion of the particles comprising the object. As the name *elasticity* implies, in this section we will discuss tension and compression of the body which are crucial to engineering design.

9.3.1 Stress

Stress denoted by the lower case Greek letter σ is a physical quantity defined as the ratio of force to cross sectional area. By this definition, the unit of stress is N/m^2 or *Pascal* in the *SI* system. Mathematically, we can write;

$$\boxed{\sigma = \frac{\mathbf{F}}{\mathbf{A}}} \tag{9.18}$$

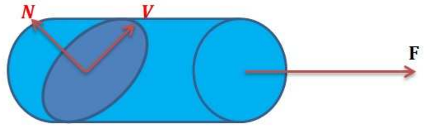

Figure 9.4: Normal and shear stresses along a failure surface due to the load **F**.

As we see from equation 9.18 stress is inversely proportional to the cross sectional area of the cylinder. This means in order to reduce stress we should consider increasing the area if weight and cost considerations are not consequential.

The physical quantity σ is usually referred to as the *Normal Stress*. However, forces acting on a body do not in general produce normal stress. Mechanical failure studies of ductile materials, such as metals usually used in engineering designs, show that the failure angle is *not* along a cross sectional area perpendicular to the direction of the force. To address this empirical observation, stress usually is divided into two components, *normal stress* normal to the failure plane as shown in figure 9.4 and the other *shear stress*, denoted by the lower case Greek letter τ (not to be confused with torque) along the plane of failure.

We therefore can write the condition for static equilibrium along the normal and the shear directions, i.e., $\sum F_N = 0$ and $\sum F_V = 0$, assuming the failure angle is θ;

$$N = F\cos\theta = \frac{\sigma A}{\cos\theta}. \tag{9.19}$$

Then the normal stress σ is;

$$\sigma = \frac{F\cos^2\theta}{A}. \tag{9.20}$$

From the trigonometric identity $\cos^2\theta = 2(1+\cos 2\theta)$, we then have;

$$\boxed{\sigma = \frac{F(1+\cos 2\theta)}{2A}}, \tag{9.21}$$

and similarly for the shear part we can write;

$$V = F\sin\theta = \frac{\tau A}{\cos\theta}, \tag{9.22}$$

or;

$$\tau = \frac{F\sin\theta\cos\theta}{A}. \tag{9.23}$$

By using the trigonometric identity $\sin 2\theta = 2 \sin\theta \cos\theta$, we then have;

$$\tau = \frac{F \sin 2\theta}{2A}. \tag{9.24}$$

It is noteworthy to emphasize that we *cannot* assign directions to stress because unlike force, stress is not a vector. Stress is also not a scalar and it belongs to a class of physical quantities called *tensors*. The discussion of this topic is well beyond the scope of this text.

9.3.2 Strain

Strain is a dimensionless physical quantity measuring the relative elongation or compression of an elastic body under stress. In figure 9.5 the elongation ΔL relative to L is the strain the cylinder is experiencing. Mathematically strain denoted by the lower case Greek letter ϵ is;

$$\boxed{\epsilon = \frac{\Delta L}{L}} \tag{9.25}$$

9.4 Modulus of Elasticity

The ratio of stress to strain is called *Modulus of Elasticity* or *Young's Modulus of Elasticity*. Denoted by the capitol Latin letter E, *Modulus of Elasticity* is a measure of the strength of material.

Mathematically,

$$E = \frac{\sigma}{\epsilon}. \tag{9.26}$$

Naturally the unit of *Modulus of Elasticity* is the same of that of stress, N/m^2 or *Pascal* in SI system.

From 9.26 we deduce that small strain translates into larger E for a given stress. For example, the *Modulus of Elasticity* for steel is 200 GPa and that of aluminum is 69 GPa. This means that steel is approximately 3 times stronger than aluminum.

We can express the modulus of elasticity in the following manner;

$$E = \frac{F/A}{\Delta L/L}, \tag{9.27}$$

or;

$$F = \frac{EA}{L}\Delta L. \tag{9.28}$$

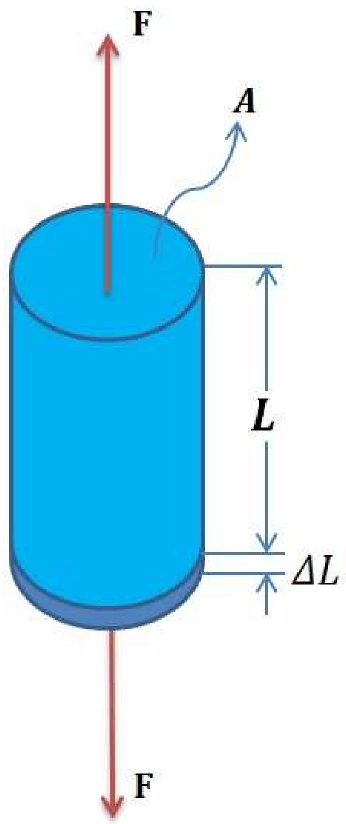

Figure 9.5: A cylinder subject to axial forces **F** at both ends and elongated by ΔL.

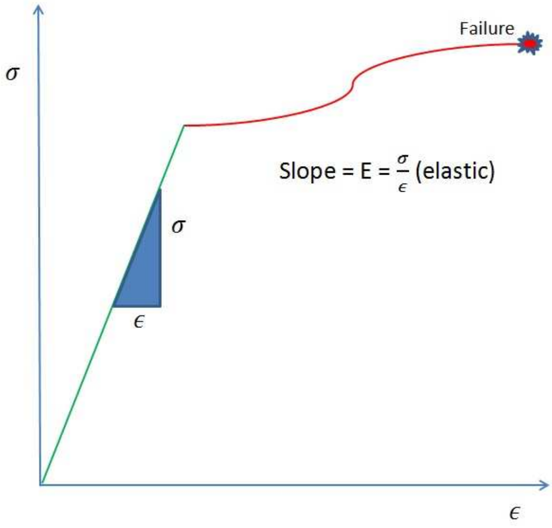

Figure 9.6: Stress-Strain curve indicating the elastic and the plastic regions of a given material experiencing mechanical stress.

The variables in the ratio $\frac{EA}{L}$ are all properties of the material used in the design and the force F is the load carried by the material and ΔL is the elongation or the compression that the material undergoes as a function of the applied force F. Note, equation 9.28 is essentially Hooke's law for a spring, where the spring constant is $k = \frac{EA}{L}$ and $x = \Delta L$.

9.5 Elasticity and Plasticity

Equation 9.26 tells us that E is the slope of the line in the strain-stress plot, i.e., $E = \frac{\sigma}{\epsilon}$. Figure 9.6 indicates two distinct regions. A linear region, shown in green, where the material undergoes an elastic deformation and after the applied load is removed, the material relaxes to its original shape. The red region is the plastic region and as shown it propagates with very little increase in the stress and if not stopped by the removal of the force it ends up with catastrophic failure.

Note, once the material undergoes plastic deformation, it would not retain its original shape.

9.6 Problems

1. An object is stationary and three forces are acting on it. If two of the forces are $\mathbf{F}_1 = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{F}_2 = 9\mathbf{i} + \mathbf{j}$, find the third force.

2. An $80 - kg$ man is standing $1.5\ m$ from one end of a small $4 - m$ long bridge. If the mass of the bridge is $500\ kg$, calculate the reaction forces at the two ends of the bridge.

3. A $60 - kg$ person is standing on top of a $200 - kg$ storage shed as shown

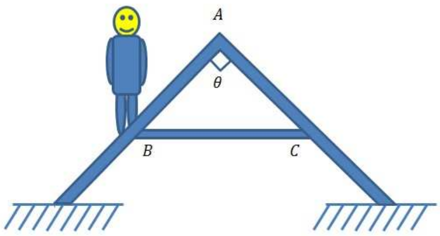

Figure 9.7: A person is standing on the roof of a storage shed.

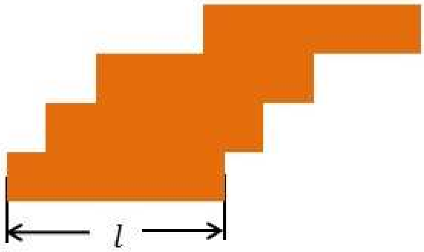

Figure 9.8: Four bricks stacked to form part of dome.

in 9.7. Find the reaction forces of the left and right supports. Assume $\theta = 90°$, $BC = 2.0\ m$ and BC is mounted half way up the shed.

4. In figure 9.7, find the tension in BC.

5. In figure 9.7, find the compression in AB and AC.

6. Four bricks, each with length l are stacked as shown in figure 9.8 to form part of a dome. Find the maximum stagger for each brick to maintain static equilibrium.

7. Figure 9.9 shows a person with mass m climbing a ladder with mass M. If the length of the ladder is l and the person is half way up the ladder and the ladder makes an angle θ with the floor, find the reaction forces on the floor and on the wall.

8. In Figure 9.9, what is the minimum coefficient of static friction of the floor in order to avoid slipping.

9. Figure 9.10 shows a ladder with mass m and length l resting on a short wall of height h. If the center of mass of the ladder is at $l/2$ and the coefficient of the static friction of the floor is μ, find the angle θ at which the ladder starts to slip. Neglect the friction between the wall and the ladder.

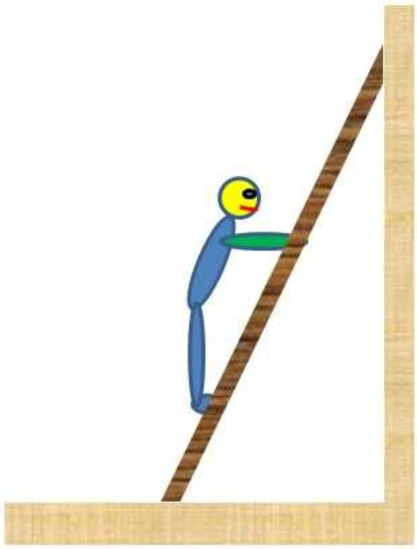

Figure 9.9: A person is climbing a ladder.

10. A mass $M = 200.0\ kg$ is hanging from two steel cables forming a triangle as shown in figure 9.11. If the unloaded angles $\alpha = 30°$ and $\beta = 60°$, and $AB = 5\ m$, find the stress and the strain in the loaded cables assuming the cables have a diameter of $4.0\ mm$. Look up the modulus of elasticity and the density of steel on the web.

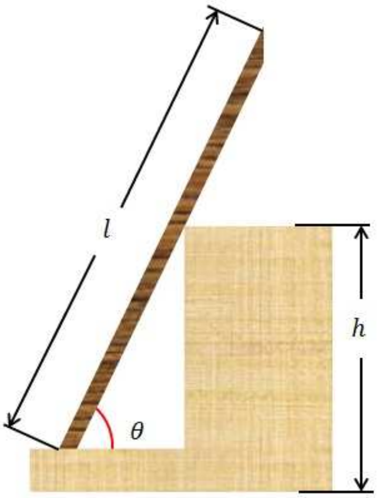

Figure 9.10: A ladder is resting on a short wall.

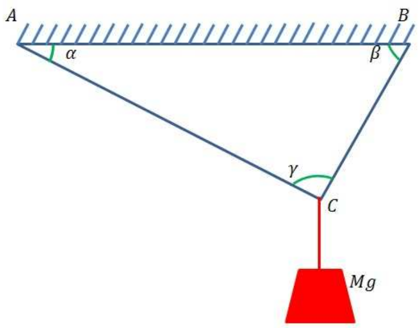

Figure 9.11: Two steel cables supporting a hanging mass.

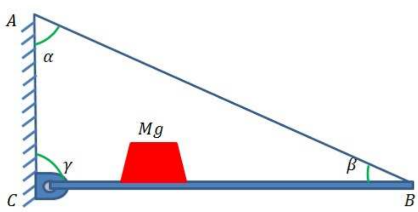

Figure 9.12: A mass is resting on a balcony.

11. In problem 10, calculate the angles after the $200 - kg$ load is attached.

12. In problem 10, calculate the minimum diameter of the cables in order to withstand the stress of the $200 - kg$ load.

13. A $100 - kg$ mass is resting on a $2 - m$ long balcony as shown in figure 9.12. Find the tension in the support steel cable AB. Calculate the reactions at A and C.

14. In problem 13, if $\alpha = 60^\circ$ and $\beta = 30^\circ$ with no load on the balcony, find the angles if the $100 - kg$ load is applied at 60.0 cm from C. Assume the diameter of the cable is 1.0 mm

15. In problem 13, if the diameter of the steel cable is 1.0 cm, find the stress in the cable assuming the $100 - kg$ load is at the edge of the Balcony, B.

16. In problem 13, find the maximum load at B that the cable can withstand without undergoing plastic deformation.

17. A steel column is 10.0 cm in diameter and 3.0 m long and is supporting a weight of 10 tons. Calculate the stress in the column. Calculate the strain in the column. How much does the column compress due to this load?

Chapter 10

Gravity

Historically the *Universal Law of Gravitation* is a rather new idea. Prior to the 17^{th} century the laws describing the motion of near Earth objects such as falling bodies and celestial objects such as the Moon, the planets, and stars were thought to be completely different. Isaac Newton was the first to unify the attraction of objects to Earth with the motion of the Moon around the Earth and show that there is one universal law governing all.

Newton published *Principia* in 1687 in which he hypothesized the inverse-square law of *Universal Gravitation.* According to Newton, two objects are attracted to each other by a force which is proportional to the product of their masses and inversely proportional to their distance. Newton, therefore, was the first to deduce that the same force which gives weight to objects here on Earth and makes them fall is also responsible for making the planets revolve around the Sun.

We can express the *Universal Law of Gravitation* in the following mathematical form.

$$\mathbf{F} = G\frac{m_1 m_2}{\mathbf{R}^2}, \tag{10.1}$$

As shown in figure 10.1 m_1 and m_2 are the masses of body 1 and 2 respectively, and R is the center to center distance between the two masses. The constant of proportionality G is the *Universal Gravitational Constant* and it is equal to $6.67384(80) \times 10^{-11}\text{N (m/kg)}^2$.

10.1 Gravity and Newton's Second Law

Recalling Newton's Second Law of motion $\mathbf{F} = m\mathbf{a}$, and the fact that we defined the weight of an object as the mass times the acceleration of gravity, we now attempt to unify Newton's Second Law with Newton's *Universal Law of Gravitation.* In order to achieve this, we equate the weight with the force of gravity between two bodies as defined in 10.1;

$$m\mathbf{g} = G\frac{mM}{\mathbf{R}^2}, \tag{10.2}$$

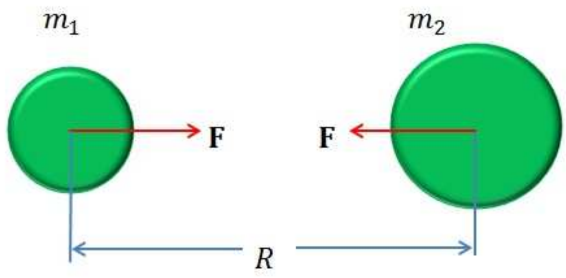

Figure 10.1: Attractive force **F** between two massive objects.

We also assume that $M \gg m$, dividing both sides of equation 10.2 by m, we then obtain the true value of acceleration of gravity.

$$\boxed{\mathbf{g} = G\frac{M}{\mathbf{R}^2}} \tag{10.3}$$

Equation 10.3 provides a mathematical tool from which we can calculate the acceleration of gravity, g, anywhere in the universe. For example, we can calculate g on the surface of a celestial body, provided we have the mass and the radius of a given body.

Example 1. Calculate the acceleration of gravity, g, on Mars. The mass and the radius of Mars are approximately $6.4 \times 10^{23}\ kg$ and 3396 km, respectively.

Answer:

To convert everything into $SI\ units$, we must convert km to m.

$$R = 3396 \times 1000 = 3.396 \times 10^6\ m \tag{10.4}$$

Using equation 10.3 and recalling the value for G, we have;

$$g = 6.67384 \times 10^{-11}\frac{6.4 \times 10^{23}}{1.153 \times 10^{13}}. \tag{10.5}$$

Hence;

$$\boxed{g \approx 3.7\, m/s}. \tag{10.6}$$

Assuming we take $g_{Earth} = 9.8\ m/s^2$, then the ratio of the weight of an object on Mars to its weight here on Earth is simply $\frac{g_{Mars}}{g_{Earth}} = \frac{3.7}{9.8} = 0.38$.

10.2 Gauss's Law of Gravity

Gauss's Law of gravity is the analog to Gauss's Law for electrostatics. Gauss's Law for static electricity is the cornerstone of electromagnetic field theory and one of the famous *Maxwell's Equations*. The definition of Gauss's Law is as follows.

Definition: *The total electric flux through a closed surface is proportional to the total electric charge enclosed within that surface.*

The mathematical representation of the above statement is;

$$\boxed{\Phi_E = \frac{Q}{\epsilon_0}}. \tag{10.7}$$

Where Φ_E is the net electric flux, Q is the total charge and ϵ_0 is called the *permittivity of free space* and it provides the inverse of the proportionality constant.

Now we can transform equation 10.7 into gravity. The analog of the electric charge Q is the mass of the body, and we therefore can write Gauss's Law for gravity.

$$\boxed{\Phi_G = CM}. \tag{10.8}$$

Here C is the constant of proportionality. The gravitational flux is defined as the field of gravity integrated over the entire closed surface.

$$CM = \oint \mathbf{g}.d\mathbf{A}. \tag{10.9}$$

Assuming a spherical surface, $d\mathbf{A} = R^2 \sin\theta d\theta d\phi$. Integrating, we therefore have;

$$CM = 4\pi R^2 \mathbf{g}a. \tag{10.10}$$

Hence, g or the gravity field becomes;

$$g = \frac{CM}{4\pi R^2}. \tag{10.11}$$

Comparing equation 10.11 with g from equation 10.3 we can find the proportionality constant C in terms of G, the universal gravity constant.

$$C = 4\pi G. \tag{10.12}$$

A note of caution here; the inverse square law for gravity is a special case applying only to spherical bodies. Gauss's Law reveals this crucial fact and it points to the fact that G, known as the universal gravitational constant has 4π associated with it which is not an integral part of it but is a consequence of spherical geometry, i.e. $A_{sphere} = 4\pi R^2$, and not a constant of nature.

Gauss's Law for gravity is therefore a more sophisticated approach to the law of gravity and reveals many aspects of the force not achievable by Newtonian approach.

10.3 Gravitational Potential

The force of gravity is a conservative force. This means that the force is derivable from the gradient of a potential. The idea of potential was not explored in chapter 7. Here we introduce this concept in terms of an analog in electric potential.

We can express the gravitational *field* **g** as;

$$\mathbf{g} = -\bigtriangledown V. \tag{10.13}$$

From equation 10.13 we can find the gravitational potential as;

$$V = -\int \mathbf{g} d\mathbf{R}. \tag{10.14}$$

The gravitational potential energy is obtained when we put a *test mass* in the gravitational potential and then we have;

$$U = mV = -m \int \mathbf{g} d\mathbf{R}. \tag{10.15}$$

For a spherical mass distribution the gravitational potential energy is;

$$U = -\frac{Gm}{R}. \tag{10.16}$$

10.4 Kepler's Laws of Planetary Motion

Kepler's Laws of planetary motion describe orbital motion of the planets around the Sun.

These three laws are:

- Planets move in an elliptical orbit around the Sun, with the Sun at one of the two foci.
- Law of conservation of areal velocities states that the area swept by a line connecting the Sun to a planet moving around the Sun is the same for equal time intervals.
- The law of periods states that the period of a planet is proportional to the square root of the cube of the semi-major axis of its orbit.

We will discuss these three laws and the equation of the orbit in the following subsections.

10.4.1 Kepler's First Law of planetary motion

The elliptical orbit of the planets or any two body system is a direct consequence of the law of conservation of mechanical energy discussed in chapter 7. This is a unique place in science where geometry and mechanics merge together to achieve the same conclusion. Here we define the ellipse.

Definition: *Ellipse is the locus of all points (P) such that the sum of their distances from two fixed points (F and F′) called focus is a constant.*

$$FP + F'P = C \tag{10.17}$$

.

Now we define conservation of energy.

Definition: *The total mechanical energy of a system remains a constant throughout the evolution of the system.*

$$T + U = E \tag{10.18}$$

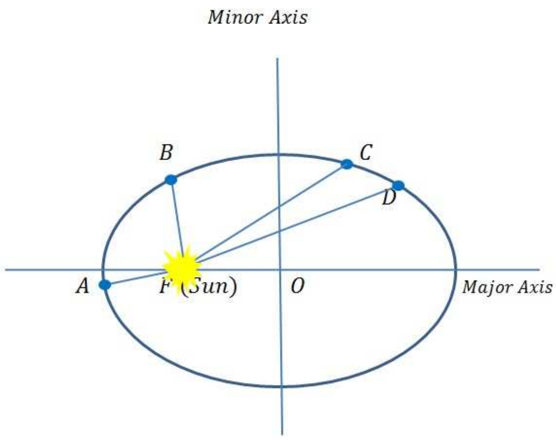

Figure 10.2: Elliptical orbit of a planet moving around a star with the star located in one of the two foci.

As we see from these two definitions, the fact that a planet moving around a star or a moon moving around a planet take an elliptical path is because it is conserving energy through its orbital motion. Note, the two distances from the two foci are proportional to the kinetic and the potential energies of the system.

$$\frac{T}{U} = \frac{F'P}{FP} \tag{10.19}$$

10.4.2 Kepler's Second Law of Planetary Motion

The law of conservation of areal velocities is a direct consequence of the law of conservation of angular momentum. We will prove this point in the following manner with the help of figure 10.2.

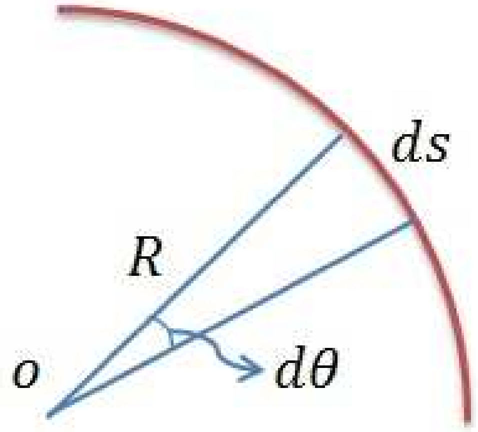

Figure 10.3: Area of a sector with an angle $d\theta$.

The angular momentum of the system is a constant throughout the time evolution of the system. We therefore can deduce that the derivative of the angular momentum with respect to time has to be equal to zero.

$$\mathbf{L} = \mathbf{R} \times \mathbf{p} \tag{10.20}$$

However, $\mathbf{p} = m\mathbf{v}$ and $\mathbf{v} = R\dot{\theta}$ and $\dot{\theta} = \frac{d\theta}{dt}$, therefore;

$$\frac{d\mathbf{L}}{dt} = mR^2\frac{d\theta}{dt} = 0. \tag{10.21}$$

As shown in figure 10.3 area dA bound by the two radii and the arc ds is $dA = \frac{1}{2}Rds$. However, $ds = Rd\theta$, therefore,

$$dA = \frac{1}{2}R^2 d\theta. \tag{10.22}$$

Comparing equations 10.21 and 10.22 we can write;

$$\frac{d\mathbf{L}}{dt} = 2m\frac{dA}{dt} = 0. \tag{10.23}$$

Hence, we can write;

$$\boxed{\frac{dA}{dt} = 0 \Rightarrow A = Constant}. \tag{10.24}$$

Equation 10.24 is exactly the statement of Kepler's Second Law and it is a direct consequence of the law of conservation of angular momentum.

10.4.3 Kepler's Third Law of Planetary Motion

Kepler's Third Law of planetary motion or the law of periods states that the square of the period of a planet orbiting around the Sun is proportional to the cube of the semi-major axis of the orbit. This statement can be derived directly from Kepler's Second Law and we will show this below.

$$\frac{dA}{dt} = \frac{1}{2}R^2\dot{\theta} = \frac{L}{2m}. \tag{10.25}$$

If we integrate over the entire period τ it yields the entire area of the ellipse, $A = \pi ab$.

We can therefore write;

$$A = \frac{L}{2m\tau} = \pi ab. \tag{10.26}$$

In an ellipse, the relation between the semi major axis a and the semi minor axis b is $b = a\sqrt{1-\epsilon^2}$ and $b = a^{1/2}\sqrt{\frac{L^2}{mk}}$ where ϵ is the eccentricity of the ellipse. From equation 10.25 we see that;

$$\tau = \frac{2m}{l}\pi a^{3/2}\sqrt{\frac{L^2}{mk}}, \tag{10.27}$$

or; with $k = GMm$ we have;

$$\boxed{\tau = 2\pi a^{3/2}\sqrt{\frac{1}{GM}}}. \tag{10.28}$$

Equation 10.28 is the statement of Kepler's Third Law.

Example 2. Calculate the period of Mars assuming the mass of the Sun $M_\odot = 1.99 \times 10^{30}\ kg$ and the distance of Mars from the Sun is 228000000 km.

Answer:

Using equation 10.28 and plugging in the value for G and keeping in mind that 1 km is 1000 m, we have;

$$\tau = 2\pi 2.28 \times 1011^{3/2}\sqrt{\frac{1}{6.67^{-11} \times 1.99 \times 10^{30}}}. \tag{10.29}$$

A year on Mars is;

$$\boxed{\tau_{Mars} = 59373505\ seconds}. \tag{10.30}$$

The year on Earth is 31536000 seconds. Hence, a year on Mars in terms of the year on Earth is;

$$\frac{\tau_{Mars}}{\tau_{Earth}} = \frac{59373505}{31536000} \approx 1.88\, years. \tag{10.31}$$

Therefore, a year on Mars is about 687 Earth days. Note, equation 10.28 has no dependence on the mass of the planet. This is because the mass of the planet is negligible compared to that of the Sun. This is not true for the case of a binary star system such as Alpha Centauri our nearest star only 4.3 light years away.

10.5 Orbits of Planets, Spaceships and Satellites

The total energy of the two-body system is $E = T + V(R)$ or $E = \frac{1}{2}mv^2 + V(R)$ and from this equation we can extract the orbital velocity of the planet $v = \sqrt{\frac{2}{m}(E - V(R))}$. We can therefore write a one dimensional equivalent potential $V' = V + \frac{L^2}{mR^2}$. Now we take the potential $V(R) = -\frac{GM}{R}$ and recognize that $\frac{L^2}{mR^2}$ is simply the centripetal force $\frac{mv^2}{R}$, then for a nearly circular orbit we can write;

$$\frac{mv^2}{R} = \frac{GmM}{R^2}. \tag{10.32}$$

Dividing by m and R on both sides of equation 10.32 we obtain the velocity of a planet around the Sun.

$$\boxed{v = \sqrt{\frac{GM}{R}}}. \tag{10.33}$$

Example 3. Find the velocity of the plant Uranus as it moves around the Sun. Assume the distance between Uranus and the Sun is 19 AU and the mass of the Sun is $1.99 \times 10^{30}\ kg$.

Answer:

One astronomical unit (AU) is the distance between the Earth and the Sun and it is $1.5 \times 10^{11}\ m$, therefore,

$$R = 1.5 \times 10^{11} \times 19 = 2.85 \times 10^{12}\ m. \tag{10.34}$$

Substituting the known values in 10.33 we obtain the velocity of Uranus.

$$v = \sqrt{\frac{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{2.85 \times 10^{12}}}. \tag{10.35}$$

Hence, we have;

$$\boxed{v = 6.8 \times 10^{3} m/s = 6.8\ km/s}. \tag{10.36}$$

As one can see this is a substantial speed even at that distance. Taking the speed of sound in the air as $\approx 340\ m/s$, the orbital velocity of Uranus is 20 times the speed of sound, or *mach* 20!

10.6 Problems

1. At what distance from the Earth is the acceleration of gravity half of that on the Earth?
2. At what distance from the Earth is the acceleration of gravity from the Earth the same as that of the Sun?
3. At what distance from the Earth is the acceleration of gravity from the Earth the same as that of the Moon?
4. If we send a spaceship toward the Moon and pass the point where the gravity of the Moon can take over, how long would it take for the ship to reach the Moon? What is its velocity when it reaches the Moon?
5. Two lead spheres each with a mass of $100.0\ kg$ are $50.0\ cm$ apart. What is the force of gravity between these two spheres?

6. Calculate the force of gravity between the proton and the electron in the H-atom. Look up all relevant parameters on the web.

7. Calculate the orbital velocity of the Moon as it revolves around the Earth.

8. Calculate the orbital velocity of Mars.

9. Calculate the orbital period of Mars.

10. Calculate the acceleration of gravity of the Earth at a depth of 1000 km.

11. Show that the acceleration of gravity of a sphere of 1.0 cm^3 of nuclear matter at 22 cm away is the same as that of the surface of the Sun! Look up the nuclear density on Wikipedia.

12. The orbital period of the Moon is 27.3 $days$ and the mass of the Earth is 5.96×10^{24} kg. Use this information to calculate the mass of the Sun.

13. The orbital period of Phobos is 459 minutes and the mass of Mars is approximately 0.11 of the mass of the Earth and its orbital period is 687 days. Use this information to calculate the mass of the Sun.

14. Compare the answers to problems 12 and 13. Which one is more accurate and why? What extra information would you need to get a more accurate result?

15. Calculate the escape velocity of the Moon and Mars.

16. A 200-kg satellite is orbiting the Earth at a altitude of 600 km. It is losing energy and goes into a spiral decaying orbit. If it loses 15000 J of energy in each orbit, calculate its distance above the Earth after 1000 orbits.

17. Find the location of the geostationary orbit for Mars.

18. Use Gauss's Law of Gravity to derive the field on an axis perpendicular to a massive disc of radius R and mass M.

19. Use Gauss's Law of Gravity to derive the field outside a long cylinder with mass m, length l, and radius R where $l \gg R$.

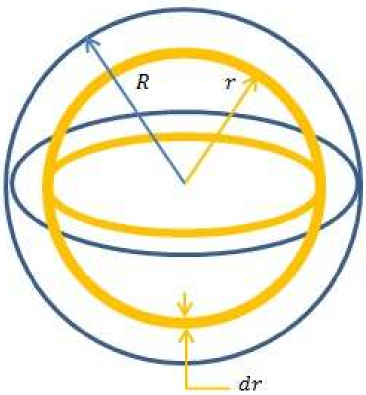

Figure 10.4: Binding energy of a sphere with radius R.

20. The gravitational self or binding energy for a spherical object is the total gravitational energy holding the sphere together. It can be thought of as separating a spherical shell and moving it to infinity. With the help of figure 10.4, prove that the binding energy of a sphere with a radius R is $U = \frac{-3GM^2}{5R}$. Using this result, calculate the binding energy of a neutron star with $M = 10M_\odot$ and $R = 10$ km.

Chapter 11

Oscillations

As the topic of this chapter *Oscillations* implies, we are to study motions that are repeating in a regular manner. Specifically, we will study oscillatory motions of the form called *simple harmonic motion*. Simple harmonic motion examples include a pendulum with a small deflection angle and a mass attached to a spring. The motion of planets and other celestial bodies can also be thought of as *oscillatory* motion, since they repeat the same kinematics and dynamical configurations continuously.

At the quantum scales of atoms, nuclei and elementary particles *oscillatory* motion is assumed as a model to describe the physical properties of these objects.

11.1 Simple Harmonic Motion

Elastic restoring force in a mass-spring system provides the best model to illustrate the idea of the *Simple Harmonic Motion*. We know from conservation of energy in chapter 6 that for a mass loaded spring system, as shown in figure 11.1, the total energy is a constant;

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E, \tag{11.1}$$

where E is the total energy of the system. From equation 11.1 the velocity of the system at any time is

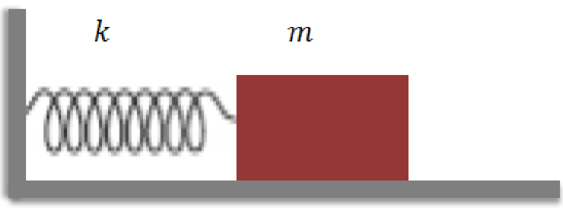

Figure 11.1: A block of mass m attached to spring oscillating back and forth.

$$v = \pm\sqrt{\frac{2E - kx^2}{m}}. \tag{11.2}$$

Note in equation 11.1 x is the actual *elongation or compression* of the spring. If we assume at time $t = 0$, the elongation (compression) is zero, i.e., the spring is relaxed, then the elastic energy of the spring is zero and we can write;

$$v_{max} = \pm\sqrt{\frac{2E}{m}}. \tag{11.3}$$

By the same logic, when the spring is stretched to its limit, the mass stops and therefore the velocity goes to zero and we can write;

$$\frac{1}{2}kx_{max}^2 = E \Rightarrow x_{max} = \pm\sqrt{\frac{2E}{k}} \tag{11.4}$$

We call this maximum deflection of the spring the *Amplitude* and denote it with the Latin letter A, therefore,

$$A = \left|\sqrt{\frac{2E}{k}}\right| \tag{11.5}$$

Recalling that $v = \frac{dx}{dt}$, we can rewrite equation 11.2 as

$$\frac{dx}{dt} = \sqrt{\frac{k}{m}}\sqrt{A^2 - x^2}. \tag{11.6}$$

Now let us separate the variables and integrate the above equation.

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{k}{m}}\int dt \tag{11.7}$$

The left side of 11.7 is the arcsin $\frac{x}{A}$. Therefore, we have a sinusoidal motion as shown in figure 11.2.

The results of the moving particle P on the trigonometry circle is shown in figure 11.3 for $0^\circ \, to \, 360^\circ$.

Now we can write the equation for the position as a function of time in the following manner;

$$x(t) = A\sin\sqrt{\frac{k}{m}}t + C. \tag{11.8}$$

Note C is the constant of integration. We denote this constant with the lower case Greek letter ϕ and it is referred to as the phase shift of the harmonic

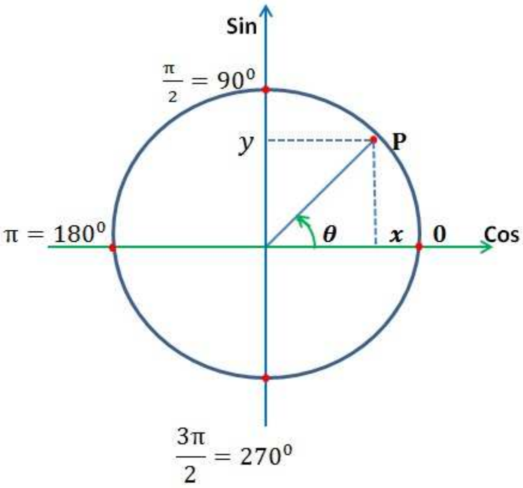

Figure 11.2: A particle moving on the trigonometry circle.

motion. As shown in figure 11.3 the cosine function lags the sine function in phase by 90°. The quantity $\sqrt{k/m}$ on the other hand is simply the angular velocity ω of the motion. Therefore, the position function for an *oscillatory* motion at any time can be written as;

$$\boxed{x(t) = A\sin(\omega t + \phi)}. \quad (11.9)$$

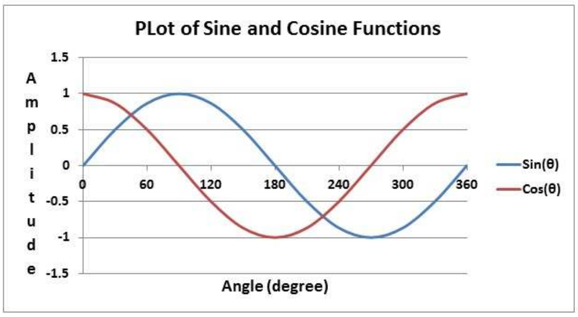

Figure 11.3: Sine and Cosine functions from $0° to 360°$.

The velocity and acceleration functions can be obtained by simply taking the consecutive derivatives of the position function and the resulting velocity function with respect to time.

$$\boxed{v(t) = A\omega\cos\omega t + \phi}, \quad (11.10)$$

and the acceleration is;

$$\boxed{a(t) = -A\omega^2\sin(\omega t + \phi)}. \quad (11.11)$$

The period τ and the frequency ν of the oscillation are $\tau = \frac{2\pi}{\omega}$ and $\nu = \frac{2\pi}{\omega}$ respectively. In terms of the mass and the spring constant we can write;

$$\boxed{\tau = 2\pi\sqrt{\frac{m}{k}}}, \quad (11.12)$$

and the frequency;

$$\boxed{\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}. \quad (11.13)$$

Example 1. Set up the equation of motion and derive its solution for the oscillations of a hanging mass spring system.

Answer:

The difference between this configuration and that of figure 11.1 is that the entire assembly is in vertical position where the weight of the mass and therefore the acceleration of gravity plays a role.

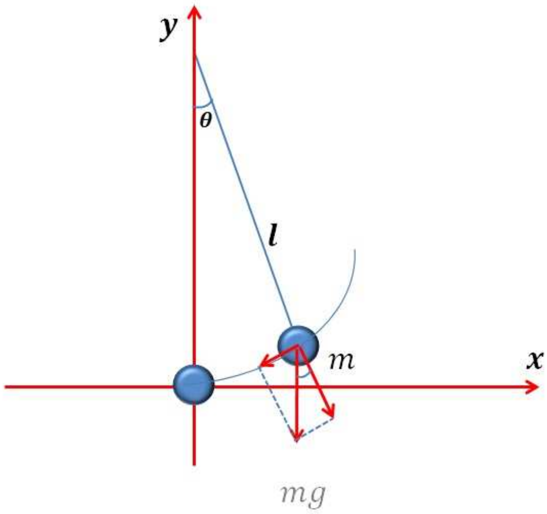

Figure 11.4: A depiction of a simple pendulum.

Hence, the equation of motion becomes,

$$M\frac{d^2y}{dt^2} + ky = Mg. \tag{11.14}$$

The difference between this differential equation and that of the horizontal spring-mass analysis is the weight Mg. The solutions are still sinusoidal and are;

$$y = A\sin(\omega t + \phi) + \frac{Mg}{k} \tag{11.15}$$

11.2 The Simple Pendulum

The study of the simple pendulum is the *epitome* of simple harmonic motions. Physically, a simple pendulum consists of a long string attached to a heavy mass usually referred to as the pendulum bob. It is assumed that the mass of the string is negligible as compared to that of the bob. As shown in figure 11.4 the weight of the bob, $m\mathbf{g}$, is always vertically downwards and has two components. The first is along the string and is balanced by the tension in the string and the second tangent to the circular path of oscillations.

We can write the sum of torques around the center of swing to be zero since this is a force-free oscillation.

$$\sum \tau = 0 \tag{11.16}$$

The torques are;

$$I\alpha + mgl\sin\theta = 0. \tag{11.17}$$

Note, the moment of inertia $I = ml^2$ and $\alpha = \frac{d^2\theta}{dt^2}$ and we therefore can write;

$$ml^2\frac{d^2\theta}{dt^2} + mgl\sin\theta = 0. \tag{11.18}$$

Dividing both sides of equation 11.18 by ml^2 we obtain the differential equation of motion for a simple pendulum.

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}. \tag{11.19}$$

Some observations of equation 11.19 are in order. First we note the absence of mass in this equation, which means the oscillation is independent of the mass of the pendulum bob. Second, we note the equation is a nonlinear second order differential equation and the integration yields an *Elliptical Integral of the first kind* which has only a series solution for the period of the form;

$$\boxed{\tau = 2\pi\sqrt{\frac{l}{g}}[1 + \frac{1}{4}\sin^2\theta_{max}/2 + \frac{9}{64}\sin^4\theta_{max}/2 + ...]}. \tag{11.20}$$

For small oscillations, i.e. small deflection angles $\sin^2\theta_{max}$ is assumed to be small and can therefore be neglected, hence the period of oscillations for a simple pendulum with small angular deflection is;

$$\boxed{\tau \approx 2\pi\sqrt{\frac{l}{g}}}. \tag{11.21}$$

Let us see how accurate equation 11.21 is. If $\theta_{max} = 15°$ then the true period is only off by 0.5%!

Example 2. If you are on a crew aboard Starship Enterprise and your shuttle craft crashes on an uncharted M-type planet and you really must know the acceleration of gravity and all your sensors are out except for a stopwatch, then what would you do?

Answer:

Find a string or a piece of wire as long as your height from the shuttle and tie it to a rock. Cut the wire in half and hold the end of the wire and deflect the rock from the vertical position by a small amount and let it swing and start the stopwatch. Count 20 complete swings and stop the watch. Let us assume you get the following measurements.

You are 180 cm tall, then half of the length of the wire is 90 cm, and let us assume you use approximately 10 cm of it to tie it around a small rock; therefore $l = 80\ cm = 0.8\ m$. The stopwatch shows 50 seconds for 20 swings which then yields a period of $50/20 = 2.5\ s$. Plugging these values in equation 11.21 provides a value for the acceleration of gravity.

$$g \approx \frac{4\pi^2 l}{\tau^2}, \tag{11.22}$$

or;

$$g \approx 5.05\ m/s^2. \tag{11.23}$$

It is noteworthy, if you are not very careful in your deflection angle and deflect even as much as 60°, the error in g is only 6.25%! Hence, this simple experiment provides a very robust method to measure the acceleration of gravity.

11.3 Problems

1. An object oscillates along the x-axis with $x = 4.0\sin(\pi t)$. Find its velocity and acceleration at $t = 2.0\ s$.
2. A horizontal mass-spring system is oscillating with a period of 0.5 s. Find its frequency and angular velocity.
3. A mass $M = 1.0\ kg$ is hanging from a spring. Another mass $m = 200\ g$ is attached to the larger mass which stretches the spring 3.0 cm further. If we remove the small mass, find the period of oscillation.

4. A mass-spring system is oscillating according to $x = 5.0\cos(\pi t + \pi/2)\ m$. Find the equations for the velocity and the acceleration of the object. Find the position, velocity and acceleration at $t = 2.0\ s$.

5. In problem 4, find a general equation for the energy of the system. Calculate its energy at $t = 2.0\ s$ if the $k = 10\ N/m$, its length is 20 cm and the mass is 500 g.

6. A $1.0 - kg$ mass is attached to a spring with a spring constant of 50 N/m. It is then displaced to the point $x = 0.2\ m$. Calculate the time it takes for the block to travel to the point $x = 10\ cm$.

7. A block of mass m on a frictionless surface is connected to two springs in series as shown in figure 11.5. Show that the frequency of oscillations is $f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$.

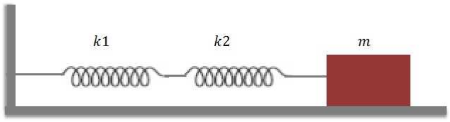

Figure 11.5: A block of mass m attached to two springs oscillating back and forth.

8. A block of mass on a frictionless surface is connected to two springs in parallel as shown in figure 11.6. Show that the period of oscillations is $T = 2\pi\sqrt{\frac{m}{k_1+k_2}}$.

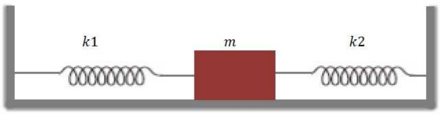

Figure 11.6: A block of mass m attached to two springs in parallel oscillating back and forth.

9. A $5.0 - kg$ wooden block is hanging from a spring with a spring constant of 600 N/m. An arrow with mass of 200 g with a velocity of 150 m/s is fired from the bottom as shown in figure 11.7 and comes to rest inside the block. Calculate the amplitude of the resulting oscillation. What fraction of the initial kinetic energy of the arrow is lost in this process?

10. A simple pendulum with a length of 100 cm is used to calculate the local acceleration of gravity. It is observed that the pendulum completes 100 oscillations in 202 s. What is the value of g on that location?

11. A race car is speeding around a race track with a constant velocity $v = 150\ km/h$. A small souvenir is hanging from a short string of length l = *10 cm* from the rear view mirror. If this simple pendulum system starts to oscillate, calculate the period of oscillation.

12. Calculate the length of a simple pendulum having a period of 2.0 s on Mars and on the Moon.

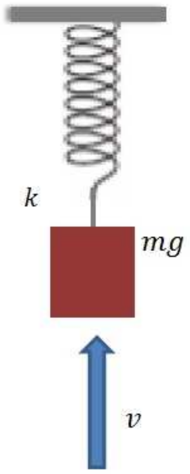

Figure 11.7: A block of mass m attached vertically to a spring with an arrow shooting straight upward and coming to rest in the block.

13. A sphere of radius R and mass M has a hole through its center which connects two points on the surface. Show that if you drop an object down this hole, the motion is a simple harmonic motion neglecting all frictions. Calculate the period for the Earth, the Moon and Mars.

14. Calculate the period of a meter-long simple pendulum at the pole and at the equator.

15. We want to design a pendulum with a period of 4.0 s that fits in a box only 1.0 m long. Can we do this? Quantify your answer!

Chapter 12

Fluid Mechanics

In this chapter, we begin to explore the behavior of flowing liquid and gases commonly known as fluids. Fluids are usually referred to as non-solid objects that flow. Sometimes there are solids such as glass or ice in glaciers that flow very slowly and yet we do not classify them as fluids.

12.1 Density

The density for a static fluid is defined, just as it was defined for a solid, as the ratio of the mass to the volume. Because fluids do not have a specific shape we must contain them in some container with known volume such as a graduated cylinder to measure its volume. Measurement of mass of fluids also must be done by using a container of known mass so the mass of the container can be subtracted. We must emphasize that the density of fluids depends on the temperature and the pressure and may vary throughout the volume. For liquids, we generally ignore this temperature and pressure, since these two factors have negligible effects on the density. However, this does not hold true for gases because the density of gases are a strong function of temperature and pressure. Mathematically density is;

$$\boxed{\rho = \frac{m}{V}}. \tag{12.1}$$

12.2 Pressure

Pressure is defined as the ratio of the normal force per unit area of a given fluid. By this definition, one may conclude that pressure is the same as *stress* in solids, but not so fast! Note, stress in general is a *tensor* with nine components because we have both normal and sheering stresses. In the case of pressure we only have one component and it is always the *normal* component and therefore pressure is a *scalar* quantity.

We can therefore write;

$$\boxed{p = \frac{F}{A} = \frac{dF}{dt}}. \tag{12.2}$$

In equation 12.2, the pressure is defined in the absence of any other external force such as gravity. The equation is only a function of the exerted force and the applied area.

The unit of pressure in SI system is the $Pascal$ (Pa) and is equal to 1.0 N/m^2. Th name Pascal for the SI unit of pressure was adopted in 1971.

In the United States a non-SI unit such as pounds per square inch is often used.

Pressure unit *atmosphere* (atm) is an established constant. It is approximately equal to typical air pressure at Earth's sea level and is defined as follows 1 $atm = 101325\ Pa = 14.69595\ lbf/in^2$.

12.2.1 Pascal's Law

Pascal's law or the Principle states that in fluid mechanics the pressure exerted at any point in a confined incompressible fluid is transmitted equally in all directions throughout the fluid. As shown in figure 12.1 the force **F** applied transmits the same pressure to all exit points shown.

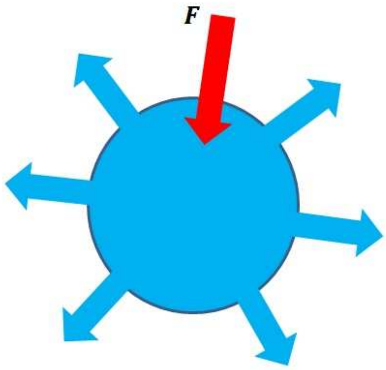

Figure 12.1: Pressure exerted by the force **F** transmits to the fluid exit points shown.

Pascal's law serves as a basis for *hydraulics* and it is a crucial component of earth moving equipment, airplanes and rockets. The law can be utilized for *hydraulics* in the following manner.

At point A we have a force $\mathbf{F_1}$ acting on an area $\mathbf{A_1}$. At any other point we would like to transmit the force $\mathbf{F_2}$ on an area $\mathbf{A_2}$. Remember that these two pressures are exactly the same according to Pascal's law. We therefore can write;

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2}, \tag{12.3}$$

or;

$$F_1 \times A_2 = F_2 \times A_1. \tag{12.4}$$

Now we can write the F_2 in terms of known force F_1 and the two known areas, $\mathbf{A_1}$ and $\mathbf{A_2}$;

$$F_2 = F_1 \times \frac{A_2}{A_1}. \tag{12.5}$$

Note, as for engineering design purposes, the factor $\frac{A_2}{A_1}$ is the controlling factor in the magnitude of the force F_2. If for example, we could increase the area A_2 by a factor of ten as compared to A_1 then the transmitted force also increases by a factor of ten! This has huge implications where substantial forces are involved. Examples for engineering applications can be cited such as braking systems for all kinds of vehicles, control systems for airplanes and rockets, power lifts and earth moving equipment and drilling.

Example 1. In a power brake system the area of the applied force is only 1% of that of the transmitted area. If the applied force is 2.0 N, calculate the transmitted force on the wheels.

Answer:

We use equation 12.5. In this equation we have;

$$\frac{A_2}{A_1} = 100. \tag{12.6}$$

Then;

$$F_2 = 100F_1, \tag{12.7}$$

substituting for F_1, we have;

$$\boxed{F_2 = 200.0\ N} \tag{12.8}$$

12.2.2 Archimedes Principle

The *Archimedes Principle* states that a fully or partially submerged body attains an upward buoyant force equal to the weight of the displaced fluid. This principle is obviously a function of the density of the fluid. We therefore can write;

$$F_{buoyant} = \rho V_{body}(-g) \tag{12.9}$$

Figure 12.2 shows the rise of the height of the fluid corresponding to the volume of the red object submerged in the liquid. An important point to note is that the buoyant force in equation 12.9 depending on the density of the fluid can actually be equal or more than the weight of the object which then can actually create an upward force.

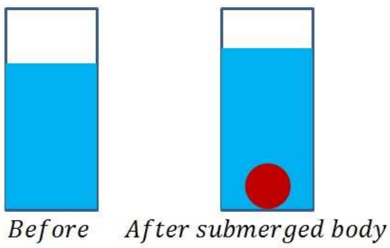

Figure 12.2: Displaced fluid as a result of a submerged object in a fluid.

Equation 12.9 now provides a tool by which we can obtain the value of g inside the fluid as a function of g in vacuum. Let us write;

$$\rho_{body}V_{body}(-g) - \rho_{fluid}V_{body}g = \rho_{body}V_{body}g_{fluid}. \tag{12.10}$$

Rearranging and dividing by V_{body} on both sides of equation 12.10, we then have;

$$g_{fluid} = g(1 - \frac{\rho_{fluid}}{\rho_{body}}). \tag{12.11}$$

Example 2. We force a small aluminum sphere into a container of mercury. Calculate g_{Hg}.

Answer:

$g_{Hg} = 13.534\ g/cm^3$ and $g_{Al} = 2.70\ g/cm^3$ and plugging these values together with $g = -9.8\ m/s^2$, we obtain;

$$\boxed{g_{Hg} = -9.8(1 - \frac{13.534}{2.70}) \approx 39.3\ m/s^2} \tag{12.12}$$

This is an upward positive acceleration inside the Hg container created for the Al sphere.

Note if the $\rho_{fluid} = \rho_{body}$ then the body experiences weightlessness inside the fluid. This physical property is used for astronaut training inside large pools of water to simulate weightlessness in space. This principle is also used for submarine navigation in oceans.

Example 3. A piece of wood with a density of $\rho_{wood} = 0.50\ g/cm^3$ is released from the bottom of a water barrel 1.00 m deep. Find the velocity of wood when it reaches the surface of the water.

Answer:

From equation 12.11, we can write;

$$g_{water} = g(1 - \frac{\rho_{water}}{\rho_{wood}}), \tag{12.13}$$

substituting the values from the ρ_{wood} and recalling that $\rho_{water} = 1.0\ g/cm^3$, we have;

$$g_{water} = -9.8(1 - \frac{1.0}{0.5} = 9.8\ m/s^2. \tag{12.14}$$

We also know from chapter 3 that;

$$v^2 = 2gh, \tag{12.15}$$

or;

$$\boxed{v \approx 4.43\ m/s}. \tag{12.16}$$

12.2.3 Pressure as Function of Depth

We have heard on television or on the Internet that if we go deep in the ocean the pressure of the water becomes unbearable and most submarines have a designed crush depth that they cannot violate for obvious safety reasons. Here we are going to understand this physical phenomena mathematically from the definition of *pressure*.

Figure 12.3 shows a dam storing water to a height h for various uses and as one can see from the figure, the bottom of the dam to the right of the figure is thicker than the top. This is an engineering design consideration because the water pressure at the bottom is much higher than the top of the reservoir as we shall see in the following derivation.

Figure 12.3: The depiction of a dam reservoir for various uses.

The pressure $p = F/A$ and we can replace the force here as the weight of the fluid, in this case water $p = W/A$. The weight $W = mg = \rho V g$ and the volume $V = Ah$. Substituting these quantities in 12.2 we have the pressure as a function of the height of the fluid or its depth.

$$\boxed{p = \rho g h} \tag{12.17}$$

We note in equation 12.17 the absence of dependence on the applied surface area. This means that the pressure at the bottom of a pool 20 meters deep is exactly the same as an ocean at the same depth. Also note the dependence of the pressure on the density ρ which is the property of the fluid. Therefore the pressure of the Earth's atmosphere corresponding to a column of air approximately 100 km high is the same as a 760 mm column of mercury! One more observation is the dependence on the acceleration of gravity. Therefore, the pressure in an ocean on Jupiter's moon, Europa with an acceleration of gravity 1.314 m/s^2 is only 0.134 of that of the Earth's oceans.

12.3 Fluid Dynamics

In this section we explore the dynamics of fluids. This means moving fluids and their properties. The flow we deal with in this chapter is a *steady* flow, the fluid is non-viscus and incompressible. In this case the quantity we like to look at and investigate is the *mass flow rate* and we discuss this idea in the following subsection.

12.3.1 Continuity Equation

The continuity of a fluid flow, for example in a pipe, is such that the flow rate at any given point is a constant. This is the statement of the *Continuity Equation*. We can derive this statement mathematically from equation 12.17. We can write pressure as F/A and considering F to be the weight of the fluid and therefore, $p = mg/A$ and

$$\frac{mg}{A} = \rho g h. \tag{12.18}$$

The gs cancel out on both sides of equation 12.18 and we take the derivative with respect to time.

$$\frac{dm}{dt} = \rho \frac{dh}{dt} A \tag{12.19}$$

The quantity $\frac{dh}{dt}$ is simply the velocity of the flow and $\frac{dm}{dt}$ which we denote with $\dot{m}$ is the mass flow rate and it is a constant. Therefore we can write the *Continuity Equation* as;

$$\boxed{\dot{m} = \rho v A = Constant} \tag{12.20}$$

12.3.2 Bernoulli's Equation

As a consequence of the *Continuity Equation*, Bernoulli's principle and the corresponding equation is of fundamental importance to fluid mechanics. The statement of Bernoulli's principle states that *for an incompressible fluid flowing in a tube, an increase (decrease) in the velocity of the flow occurs simultaneously with a decrease (increase) in pressure or a decrease (increase) in the fluid's potential energy.*

We will derive Bernoulli's Equation using the laws of conservation of energy. As shown in figure 12.4 the shaded blue elements are the amount of the mass flow rate at the two cross sections 1 and 2.

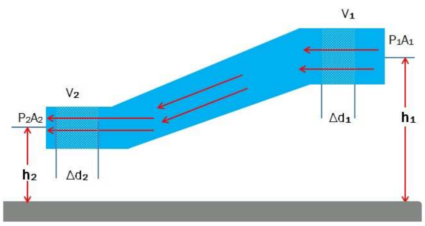

Figure 12.4: A depiction of fluid flow in a tube with varying cross sections.

The work done by the two shaded elements minus the gravitational potential energy is;

$$W = p_1 A_1 \Delta d_1 - p_2 A_2 \Delta d_2 - mg(h_1 - h_2). \tag{12.21}$$

Recall the fluid is incompressible and therefore the density ρ is a constant and the volumes of the fluid in 1 and 2 are the same; then we can write;

$$W = \frac{m}{\rho}(p_1 - p_2) - mg(h_1 - h_2). \tag{12.22}$$

Since the velocities of the two shaded elements at 1 and 2 are different and have to compensate for the decrease in the tube cross section as the *Continuity Equation* requires, then the work done is simply the difference of the kinetic energies at 1 and 2, hence;

$$W = T_1 - T_2 = \frac{m}{2}(v_2^2 - v_1^2). \tag{12.23}$$

Equating the right hand sides of the equations 12.22 and 12.23 we obtain;

$$\frac{m}{\rho}(p_1 - p_2) - mg(h_1 - h_2) = \frac{m}{2}(v_2^2 - v_1^2). \tag{12.24}$$

Canceling the mass m and multiplying through by the density ρ, we get;

$$p_1 + \frac{\rho}{2}v_1^2 + \rho g h_1 = p_2 + \frac{\rho}{2}v_2^2 + \rho g h_2. \tag{12.25}$$

Hence, Bernoulli's equation can be written as;

$$\boxed{p + \frac{\rho}{2}v^2 + \rho g h = Constant}. \tag{12.26}$$

12.3.3 Applications of Bernoulli's Equation

The two most important applications of the Bernoulli's equation are the aerodynamic lift and rocket thrust. Shown in figure 12.5 is the cross section of a typical airplane wing. According to the foil design the air velocity on the top of the wing is faster than the bottom and therefore according to the Bernoulli's equation, this mismatch in the air flow velocities creates an upward pressure difference.

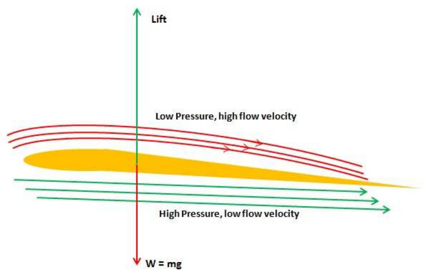

Figure 12.5: A typical cross section of an airplane wing with air flow of higher velocity on the top than bottom.

The pressure difference then multiplies by the area of the wing and provides the *lift* force for the aircraft.

When you fly on a plane and have a window seat, you may have noticed that the captain at take off and landing extends the so-called *flaps*. This is done in order to increase the surface area of the wings and thereby increase the *lift* of the plane.

The second important application of the *Bernoulli's Equation* is the rocket propulsion system. As shown in figure 12.6 the combustion chamber shown in yellow has a cross sectional area of A_1 and it is at a pressure p_1 and the exhaust has an orifice with a cross sectional area of A_2 and has a pressure p_2. According to the *Bernoulli's Equation* we can write;

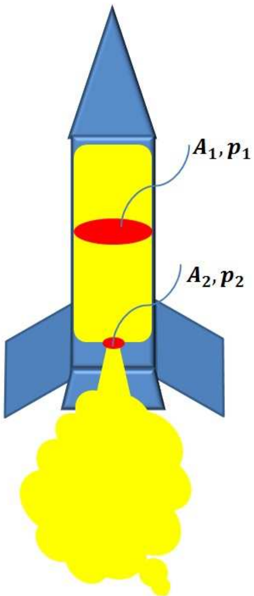

Figure 12.6: A depiction of the rocket thrust system with much smaller orifice area allowing a much higher pressure than the combustion chamber.

$$p_1 - p_2 = \frac{\rho}{2}(v_1^2 - v_2^2). \tag{12.27}$$

Note, we do have the part of the *Bernoulli's Equation* due to gravity because we assume that the rocket is propelled in outer space and therefore it experiences weightlessness. Therefore, the exhaust velocity is;

$$v_2^2 = \frac{2(p_1 - p_2)}{\rho} - v_1^2. \tag{12.28}$$

Now we invoke the *Continuity Equation* and we can therefore write;

$$A_1 v_1 = A_2 v_2. \tag{12.29}$$

Assuming the orifice cross sectional area is much smaller than the combustion area A_1, therefore according to equation 12.29 $v_1 \ll v_2$. So this allows us to neglect the effects of v_1^2 and therefore we have;

$$\boxed{v_2^2 = \sqrt{\frac{2(p_1 - p_2)}{\rho}}}. \tag{12.30}$$

12.4 Problems

1. Calculate the force experienced in an area of 10.0 cm^2 at the bottom of a water tank at a depth of 5.0 m.
2. Find the magnitude and direction of the hydrostatic force on the side of a water tank at a depth of 150.0 cm.
3. A hydraulic jack for a car has a diameter of 20.0 cm. What pressure is required to lift a $1 - ton$ car?
4. A $2.7 - g$ ping pong ball is held at the bottom of a bucket of water. If the density of the ball is 0.084 g/cm^3, find the force required to keep the ball under water.
5. If we let the ping ball in problem 3 be released from a depth of 20.0 cm under the water, find its velocity as it emerges from the water.
6. A steel ball bearing with a density of 8.05 g/cm^3 is released from the surface of water in a barrel 60.0 cm deep. Find the velocity of the ball bearing when it hits the bottom of the barrel.
7. Calculate the smallest area required for a block of ice 50.0 cm thick to support a $60 - kg$ person on seawater. Look up the densities of seawater and ice on the web.
8. A small spring-loaded gun fires a small spherical wooden object horizontally from the bottom of a 30.0 cm deep bucket of water resting on a horizontal floor. If the velocity of the wooden sphere is 10.0 m/s, find the minimum distance of the gun from the side of the bucket in order for the wooden sphere to clear the bucket. Density of wood is 0.38 g/cm^3.
9. In problem 8, find the velocity and the angle of the sphere as it enters the air. How far would it travel before it hits the floor?
10. A cylinder contains both water and propylene. Because propylene has a lower density than water it floats on the top. A wooden cubical block, 8.0 cm on a side, is then dropped in the cylinder and comes to rest with 2.0 cm inside the water and the rest in the propylene. Find the density of the block. The density of propylene is 0.5 g/cm^3.
11. What must be the pressure in a fire hydrant for the water to reach a height of 20.0 m?
12. A U-tube is filled with water. As shown in figure 12.7, if we pour another liquid in the U-tube that does not dissolve in water and the water rises to a height h find the density of the liquid.
13. On May 8 1654, Otto von Guericke the inventor of the vacuum pump demonstrated it in front of the Imperial Diet and the Emperor Ferdinand III in Regensburg. Two teams of 15 horses each could not pull apart two hemispheres held together to make a vacuum sphere. Show that the total force holding the two hemispheres together is $F = \pi R^2 P$.

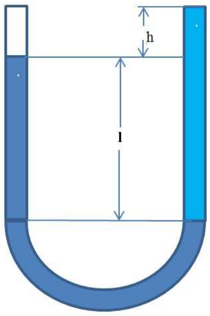

Figure 12.7: A U-tube filled with water and another liquid.

14. A storage tank 10.0 m deep has a relief valve at the bottom. If we open the valve, what is the pressure that the valve experiences?

15. A water pipe has a diameter of 3.0 cm at ground level. If the flow rate has a velocity of 1 m/s and a pressure of 88 kPa and the pipe narrows to 1.5 cm diameter, calculate the water pressure and its velocity 10.0 m above the ground.

16. An upright cylindrical tank of water with a height H has a valve at a height h below the top of the water in the tank. If we open the valve, show that the velocity of the water is $v = \sqrt{2gh}$.

17. The wing of an airplane has a mass of 300.0 kg. The air streams on the top of the wing at a speed of 100.0 m/s and at the bottom 70.0 m/s. If the area of the wing is 6 m^2, calculate the net force on the wing.

18. Calculate the thrust of a rocket with an nozzle diameter of 20.0 cm and a pressure difference of 5000.0 kPa.

Chapter 13

Wave Mechanics

If you drop a pebble on the surface of a pond on a calm day when the water is still, you would see waves generated in a circular fashion originating from the place the pebble was dropped. This is really the classic example for the study of waves and hence the topic of wave mechanics in physics. The wave aspect of motion in physics dominates travel of sound and light and other microscopic objects such as molecules, atoms, nuclei and elementary particles.

13.1 Longitudinal and Transverse Waves

Longitudinal waves are waves that their displacement is parallel to the direction of their propagation. Any pressure or stress wave is a longitudinal wave. Sound waves to be discussed later in the next chapter are good examples of longitudinal waves.

Transverse waves are waves that their displacement is perpendicular to the direction of their propagation. As mentioned above a ripple in a pond of still water or a wave on a string are examples of transverse waves. Another excellent example of transverse waves is the electromagnetic waves, such as visible light, radio waves, microwaves, $x - rays$ etc.

Both the Longitudinal and Transverse Waves can be represented by the sinusoidal functions described in chapter 11.

13.2 Frequency, Period and Wavelength

We defined these three topics in chapter 11 but here we redefine them for waves. The *frequency* of a wave be it either longitudinal or transverse is the number of occurrences of the wave repeating its original configuration per unit time. Therefore for a sine wave the frequency of the wave is a measure of how many times the sine wave is repeated per unit time. The unit of frequency, denoted with the lower case Greek letter ν, in the SI system is *cycle/s* or *Hertz*.

The *period* of a wave is the time it takes to complete a full cycle. Therefore, the *period* is inversely proportional to the *frequency*. The unit of the *period*, denoted with the lower case Greek letter τ, is *second* (s).

We can therefore write;

$$\boxed{\tau = \frac{1}{\nu}}. \tag{13.1}$$

The wavelength is the length of a complete cycle in a wave. The unit of the wavelength, denoted with lower case Greek letter λ, in the SI system is *meter* (m).

13.3 Wave Velocity

The velocity or the speed of a wave is a measure of how fast the wave propagates through a medium. The velocity as it was defined in chapter 3 is;

$$v = \frac{x}{t}. \tag{13.2}$$

Therefore, we can define the speed of the wave as the ratio of the wavelength to the period. So we can write;

$$\boxed{v = \frac{\lambda}{\tau}}. \tag{13.3}$$

Another way to write the speed of a wave is in terms of its frequency;

$$\boxed{v = \nu\lambda}. \tag{13.4}$$

13.4 Vibrating String

In order to visualize the wave motion, the *Vibrating String* provides the classic example and best lecture demonstration. Figure 13.1 shows three modes of vibration of a string.

In this figure 13.1, we have displayed three subfigures showing a string under tension T with three modes of vibration. The top plot shows the so-called *first harmonic* or the *fundamental frequency* with the two *nodes* at both ends fixed and one *antinode* in the middle. The wavelength λ is twice the length of the string L.

The middle plot in figure 13.1 shows the second harmonic where we have a complete sine wave and hence the complete wavelength making λ equal to the length of the string L.

The bottom plot in figure 13.1 shows the third harmonic where we have 1.5 sine waves and hence 1.5 λ making λ equal to 2/3 of the length of the string L.

These modes of string vibration depends on the tension T of the string. Figure 13.2 shows the vector $\mathbf{T}$ and the radius of curvature R and we assume the linear mass density of the string is μ. We therefore can write;

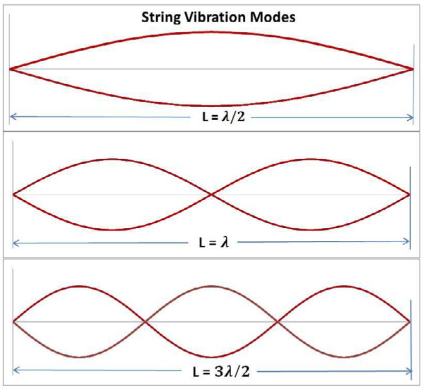

Figure 13.1: Vibration modes of a string under tension.

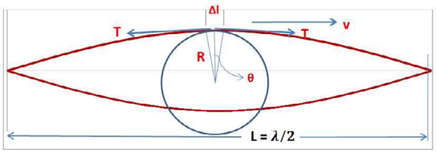

Figure 13.2: A force analysis of a string under tension.

$$2T\sin\theta = 2T\frac{\Delta l/2}{R}. \qquad (13.5)$$

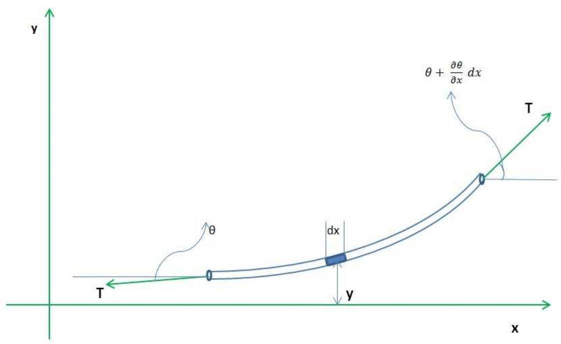

Figure 13.3: A segment of the string vibrating in the $y - direction$.

Assuming θ is small, then $\sin\theta \approx \theta$, hence;

$$2T\theta = T\frac{\Delta l}{R}. \tag{13.6}$$

As the string vibrates every molecule in the string experiences a centripetal force with respect to the center of the circle shown in 13.2. The element of the string Δl has a mass of $\mu \times \Delta l$; we therefore can write the tension T in the string as;

$$T = \frac{\mu \Delta l v^2}{R}. \tag{13.7}$$

Now we equate the two tensions and we obtain;

$$T\frac{\Delta l}{R} = \frac{\mu \Delta l v^2}{R}. \tag{13.8}$$

Then the wave velocity can be written as;

$$\boxed{v = \sqrt{\frac{T}{\mu}}}. \tag{13.9}$$

13.5 The Wave Equation

In deriving the wave equation we refer to figure 13.3 and it is important to mention that we study the vibration of the string only in the $y - direction$. Note we can write the component of the T in the $y - direction$ as $T\sin\theta$. But we realize that θ is small and therefore we can replace $\sin\theta$ with θ in $radians$. From Newton's Second Law $\mathbf{F} = m\mathbf{a}$ we can write a as;

$$\mathbf{a} = \frac{\partial^2 y}{\partial t^2}. \tag{13.10}$$

The force then is an infinitesimal difference between the two ends of the string segment and that is,

$$F_y = T\frac{\partial\theta}{\partial x}dx. \tag{13.11}$$

Therefore, we can write;

$$\frac{\partial\theta}{\partial x} = \frac{\partial^2 y}{\partial t^2}. \tag{13.12}$$

Recall the slope $\tan\theta = \frac{\partial y}{\partial x}$. However, since θ is small then $\theta = \frac{\partial y}{\partial x}$, we can then write;

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}. \tag{13.13}$$

Using equation 13.9 we see that the ratio $\frac{\mu}{T}$ is simply the inverse of the square of the wave velocity, v^2. Therefore, we have;

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}. \tag{13.14}$$

Equation 13.14 is the *Classical Wave Equation* and it is applicable, for example, to the propagation of sound in the air. It is a second order partial differential equation and we discuss its solutions in the following section.

13.6 Solution to the Wave Equation

The solution to the wave equation for the string of figure 13.1 can have the form of a sinusoidal wave;

$$y = A\sin(\omega t + \phi). \tag{13.15}$$

In equation 13.15 A is the amplitude which refers to the maximum y deflection or the y value corresponding to the antinode, $\omega = 2\pi\nu = \frac{2\pi}{\tau}$ and ϕ is the phase shift. We can also write this in terms of wavelength, wave velocity v and the displacement x;

$$\boxed{y = A\sin\frac{2\pi}{\lambda}(x \pm vt)}. \tag{13.16}$$

Note, λ is the wavelength and v is the wave velocity. The solution of equation 13.16 which is the solution of the wave equation 13.14 can be verified by direct substitution.

13.7 Power and Intensity of Waves

A very important physical quantity associated with waves is the power they generate as they travel. This also can be thought of as the intensity of a wave along its path. We experience this phenomenon in our everyday life. When we are close to a source of sound it sounds louder and when we are far away it sounds quieter. This is a measure of the amplitude or the intensity of the source at a given distance. This is also true in the case of a source of light. As we get closer to a source of light it gets brighter and as we get farther from the source it gets dimmer. We now derive the intensity or the power of a wave. The kinetic energy carried by a vibrating string can be written as;

$$dT = \frac{1}{2}dm(\frac{dy}{dt})^2_{max}. \tag{13.17}$$

Power is simply the rate of expenditure of energy and for a full oscillation or cycle the time is the period of oscillation, τ. We can also replace the mass differential dm with the linear density of the string times its wavelength λ. We therefore can write;

$$P = \frac{1}{2}\mu\lambda(\frac{dy}{dt})^2_{max}/\tau. \tag{13.18}$$

However, $\frac{\lambda}{\tau}$ is simply the wave velocity, hence we have;

$$P = \frac{1}{2}vi\mu(\frac{dy}{dt})^2_{max}. \tag{13.19}$$

From the general solution of the wave equation, we can write $(\frac{dy}{dt})_{max} = \omega y_m$ and then;

$$P = \frac{1}{2} v\omega {y_m}^2 \tag{13.20}$$

The intensity of an isotropic source of wave is the power transmitted across a unit area perpendicular to the direction of the wave.

$$P = 4\pi R^2 I. \tag{13.21}$$

Equation 13.21 tells us that the intensity drops as a function of the square of the distance from an isotropic source of wave.

13.8 Interference, Standing Waves and Resonances

In wave mechanics, the physical phenomenon known as interference is the superposition of two waves forming a resultant wave of greater or lower amplitude. Interference can be either *constructive* or *destructive* which results in a raising or a lowering of the amplitude, respectively. Interfering waves must come from the same source and they must have nearly the same frequency. They also must have different phase shifts. Note interfering waves do not have to have the same amplitude. Interference can occur with all types of waves, for example, light and sound waves.

We can describe interference of waves mathematically as follows;

$$y_1 = A\sin(kx - \omega t - \phi), \tag{13.22}$$

and;

$$y_2 = A\sin(kx - \omega t). \tag{13.23}$$

The addition of these two waves yields the result of the *interference* between the two waves y_1 and y_2.

$$y = y_1 + y_2 = A[\sin(kx - \omega t - \phi) + \sin(kx - \omega t)]. \tag{13.24}$$

Using the trig identity,

$$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}, \tag{13.25}$$

we then have,

$$y = 2A[\cos(\phi/2)]\sin(kx - \omega t - \phi/2). \tag{13.26}$$

Equation 13.26 describes a wave resulting from the interference of the two waves y_1 and y_2. Study of this equation tells us that the amplitude $2A[\cos(\phi/2)]$ depends on the phase shift ϕ. If ϕ is small then the amplitude of the two waves becomes nearly equal to $2A$ and it implies that the two waves are in phase and that they interfere *constructively*. On the other hand if $\phi \approx 180°$, then $\phi/2 \approx 90°$ and we know that $\cos 90° = 0$ and therefore the two waves interfere *destructively*.

Standing waves or stationary waves are waves that remain stationary as a function of time. This phenomenon can occur because the two waves interfere as they travel in the opposite direction of one another. The best example is a string fixed at both ends.

Mathematical interpretations of the above paragraph are described below. We have two waves with identical frequencies as follows,

$$y_1 = A\sin(\omega t + kx), \tag{13.27}$$

and;

$$y_2 = -A\sin(\omega t - kx). \tag{13.28}$$

Note, these two waves are out of phase by $2kx$, i.e. kx changes to $-kx$. The amplitudes are opposite as A changes to $-A$. Let us add these two trig functions;

$$y = y_1 + y_2 = A[\sin(\omega t + kx) - \sin(\omega t - kx)]. \tag{13.29}$$

Using the trig identity of equation 13.25, we have;

$$\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}, \tag{13.30}$$

we can then write;

$$\boxed{y = [-2A\cos\omega t]\sin kx}. \tag{13.31}$$

In the string with both ends fixed, equation 13.31 describes a wave with a time-varying amplitude $-2A\cos\omega t$ shown in brackets above. This gives the positional amplitude of standing waves in the string.

13.9 Problems

1. Calculate the speed of a wave if the period is 2 ms and the wavelength is 50 cm.

2. Calculate the frequency and angular velocity of a wave with a period of 2 μs.

3. Find the wavelength of a wave with frequency of 500 Hz and a velocity of 300 m/s.

4. A string is under a tension of 100 N. If the density of the string, made of brass, is 8.4 g/cm^3 and the diameter of the string is 500 $\mu(m)$, find the velocity of the wave in the string.

5. The general solution to the classical wave equation is $y = A \sin \frac{2\pi}{\lambda}(x \pm vt)$. Write the equation in terms of the wave number. Find the maximum displacement of the oscillation.

6. What is the speed of a transverse wave in a steel cable under a tension of 100 N with a mass of 10 kg and a length of 2 m?

7. The equation of a wave in a string is $y = 2.0\ cm \sin(150\ m^{-1}x - 550\ t^{-1})$. Calculate the wave speed and the linear density of the string.

8. The speed of a wave in a string is 200 m/s and the tension is 200 N. Find its linear density. If the speed changes to 230 m/s, what would be the tension in the string?

9. A circular hoop is rotating in interstellar space around an axis perpendicular to the plane through its center. If the tangential velocity is $\mathbf{u}$, find the speed of the wave in the hoop. Note, you do not need the radius of the hoop.

10. Find the intensity of a source of $40 - W$ light at a distance of 10 m.

11. Find the power of a vibrating string with a frequency of 50 Hz and has a wave velocity 200 m/s with a linear density of 10.0 g/m. The maximum deflection of the string is 2.0 cm.

12. Supernova 1987A was detected by three neutrino detectors, Kamiokande, IMB and Baksan in 1987, 2 to 3 hours before the light from the explosion was seen. The star Sanduleak $-69°$ 202 in the Large Magellanic Cloud went supernova 167000 years ago. If the number of neutrinos from this event was $\approx 10^{58}$ neutrinos in $\approx 13\ s$ calculate the flux of neutrinos here on Earth per square centimeter.

13. In problem 12, calculate the total power received by the Earth due to these neutrinos if the average kinetic energy of the anti-neutrino was 15 MeV.

14. A string has a tension of 10.0 N and a mass of 200.0 g and a length of 1.3 m. What is the speed of the wave? Find the lowest resonance frequency of the string.

Chapter 14

Sound

We use the discussions and derivations in previous chapters as a segue to study the important topic of sound. In our everyday experience we hear sounds of various types and listen to music and other acoustical sources. We will study various physical quantities of acoustics and sound in the following sections.

14.1 Sound Waves

Sound waves are longitudinal waves or pressure waves. One also should note that sound waves propagate from a source in all directions in an isotropic manner. Here for simplicity we only study the propagation of sound in one direction, namely the $x - direction$.

It is a common misunderstanding that when we talk about sound, the perception leads us to the erroneous conclusion that we are referring to *audible* sound. The fact is that the range of audible frequencies to an average human with healthy hearing is from 20 Hz to 20000 Hz, although the range of frequencies individuals hear is greatly influenced by environmental factors. As we age the upper high frequency of 20000 Hz is very much affected before the lower frequency sounds. The high frequency limit varies significantly among different species, for example, dogs can hear up to 45000 Hz and cats up to 64000 Hz and finally bats use various ultrasonic ranging (echolocation) techniques to locate their prey which can be up to 200000 Hz!

Ultrasound of 100 kHz and above are used as a diagnostic tool in medicine and as detection and navigation tools in ships and submarines.

14.2 Speed of Sound

As a historical note, the first analytical calculation of the speed of sound was performed by Sir Isaac Newton in Proposition 49 of Book II of the Principia. He calculated a speed of sound of 979 ft/s at sea level which was too low by 137 ft/sec from the actual measured value of 1116 ft/sec.

To determine the speed of sound, we refer to figure 14.1. In this figure a piston oscillates back and forth in an air-filled tube and the densely peppered regions shown are the longitudinal sound waves propagating in a pipe of cross sectional area A. We assume the tube has infinite length and therefore the reflection of the sound can be ignored. The dense area has a higher pressure than the adjacent light shaded area, therefore, $dp = p_1 - p_2$.

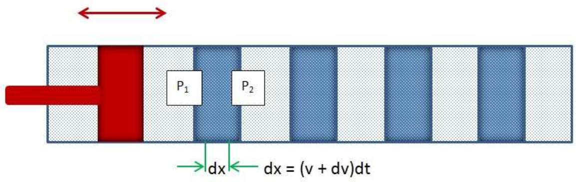

Figure 14.1: Sound waves generated by an oscillating piston in an infinitely long air-filled tube. The densely peppered areas are where the air is compressed due to longitudinal sound waves creating pressurized regions in the tube.

We can therefore write;

$$F = dpA. \tag{14.1}$$

We know that Newton's second law is;

$$\mathbf{F} = m\mathbf{a}. \tag{14.2}$$

Now we have to express both the mass m and the acceleration $\mathbf{a}$ in terms of parameters available from the gas or the air in this case inside the tube. We have;

$$m = \rho V = \rho dxA. \tag{14.3}$$

However, $dx = vdt + dvdt$ and we can neglect $dvdt$ and recalling that $a = dv/dt$, we can therefore write;

$$F = \rho vdtA\frac{dv}{dt}, \tag{14.4}$$

canceling dt's and equating the right sides of equations 14.1 and 14.4 we have;

$$dp = \rho vdv. \tag{14.5}$$

Rearranging 14.5 and multiplying both sides by v, we get;

$$\rho v^2 = \frac{dp}{dv/v}. \tag{14.6}$$

Now, recognizing that the fractional velocity $\frac{dv}{v}$ is the same as the fractional volume which is a measure of the compression of the gas due to the pressure in the sound waves, then we can write;

$$\frac{dV}{V} = \frac{\rho v dt A}{\rho A v dt} = \frac{dv}{v}, \tag{14.7}$$

we therefore have;

$$\rho v^2 = \frac{dp}{dV/V}. \tag{14.8}$$

The quantity $\frac{dp}{dV/V}$ is a property of the gas or the medium transmitting the sound waves and is called the *Bulk Modulus* and we denote it with the capital Latin letter B. Therefore the sound velocity is;

$$\boxed{v = \sqrt{\frac{B}{\rho}}}. \tag{14.9}$$

From the mathematical definition of the *Bulk Modulus* we can then define this property of matter as;

Definition: *Bulk Modulus is the property of matter, solid, liquid or gas which shows a change in volume due to an applied external pressure.*

Note the units of the *Bulk Modulus* is the same as the unit for pressure and in the SI system it is therefore *Pascal*.

Example 1. Find the *Bulk Modulus* for water ice if the density of ice is $0.911\ g/cm^3$ and the speed of sound in ice is $3950\ m/s$.

Answer:

We have;

$$v = \sqrt{\frac{B}{\rho}}. \tag{14.10}$$

Rearranging 14.10 we can write B as;

$$B = \rho v^2. \tag{14.11}$$

Now we must get all of our units converted into SI units. The speed of sound is already in SI, but the density is not, we therefore can write;

$$\rho = 0.911\ g/cm^3 = 911\ kg/m^3 \tag{14.12}$$

Therefore B for ice is;

$$\boxed{B = 911 \times 3950^2 \approx 1.42 \times 10^{10}\ Pa = 14.2\ GPa}. \tag{14.13}$$

This tells us that ice is not very compressible as we would intuitively expect.

14.3 Intensity of Sound

The intensity of sound or acoustic intensity is defined as the average flow rate of sound energy through a unit area normal to the propagation of the sound wave. The unit of sound intensity is $Watt/m^2$. Note, $Watt$ is the unit of power in the SI system.

In order to mathematically formulate the intensity of sound, we define a quantity ε as the energy density per unit volume of space. Then we can write the flow rate of energy, dE/dt as;

$$\frac{dE}{dt} = \varepsilon v, \tag{14.14}$$

and we therefore define the intensity;

$$\boxed{I = \frac{dE}{dt}}, \tag{14.15}$$

or;

$$\boxed{I = \varepsilon v}. \tag{14.16}$$

It can be shown that the energy density is $\varepsilon = \frac{\rho\omega^2 A^2}{2}$ and then the alternative form of sound intensity is;

$$\boxed{I = \frac{\rho v \omega^2 A^2}{2}}. \tag{14.17}$$

The sound level of intensity is defined as;

$$L = 10 \log_{10} \frac{I}{I_0}. \tag{14.18}$$

In equation 14.18 L is the *intensity level* and it refers to the *loudness* of the sound. I_0 is the faintest sound audible by an average human and it is 10^{-12} W/cm^2. The unit of L is *Decibel* and is shown by its abbreviation (db). For example a sound of 120 db is painful to the ear and could cause hearing loss.

14.4 Beat

Beat, in acoustics, is manifested when an interference between two sound waves of slightly different frequencies occurs. It is a periodic variation in amplitude whose rate of occurrence corresponds to the difference of the two frequencies.

In musical instruments, beats can be heard easily by the listener. Instruments can be tuned to produce two different tones of nearly identical frequencies, the difference in two frequencies will generate the beat.

Mathematically, the beat between two sound waves of slightly different frequencies is shown in figure 14.2.

$$\cos(2\pi\nu_1) + \cos(2\pi\nu_2) \tag{14.19}$$

In equation 14.19 we have set $\nu_1 = 1.1\nu_2$ or a 10% difference in the two frequencies. This is the assumption we made to plot figure 14.2.

14.5 The Doppler Effect

In acoustics, when a source of sound is moving toward a stationary or moving listener, the pitch or the frequency of the sound changes as a function of their

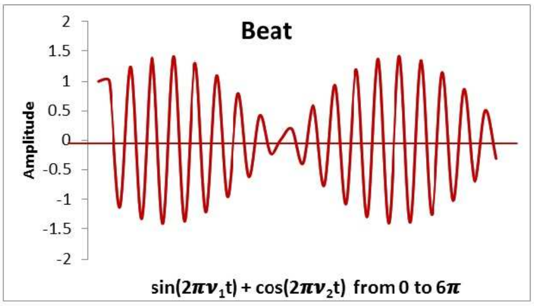

Figure 14.2: Two sound waves producing beat when their frequencies are different by 10%.

relative velocities. This is known as the *Doppler Effect or Doppler Shift* named after the Austrian physicist Christian Doppler who proposed it in 1842.

We experience the Doppler Effect in our everyday life. For example when we hear the siren of an ambulance speeding toward us, we hear a high pitch and when it is speeding away from us, we hear a lower pitch.

We can work out the mathematics of the Doppler shift according to what we learned in the previous chapter regarding wavelength, frequency and speed of a wave. We know that;

$$\lambda = \frac{v}{\nu}. \tag{14.20}$$

Now we add the effect of the moving source of sound (denoted as v_s) on to the wavelength which the listener hears,

$$\lambda_L = \frac{v - v_s}{\nu}. \tag{14.21}$$

Equation 14.21 is for a source or a listener moving toward one another. Note, λ_L is the wavelength of the sound the listener hears. If they are moving away then we have;

$$\lambda_L = \frac{v + v_s}{\nu}. \tag{14.22}$$

In both equations 14.21 and 14.22 there is a change in the wavelengths which means that the frequency or the pitch of the sound also changes.

The Doppler shift is also used in submarines for targeting and localizing other submarines. It is used to calculate the velocity of a submarine or other sea-borne vessel using both passive and active sonar systems.

We should emphasize that the Doppler shift works for all types of waves including electromagnetic radiation and light. The Doppler shift toward the *red* part of the visible light or the so-called *Red Shift* is a yardstick by which astronomers calculate the rate of the expansion of the universe.

Another electromagnetic radiation application of the Doppler shift is the Radar. Radar are radio waves used to find the velocity of a given object such as planes or automobiles. Traffic police also use the radar to issue tickets to speeding motorists by measuring the speed of cars using the Doppler effect.

14.6 Problems

1. Calculate how long it would take for sound to travel around the Earth. The radius of the Earth is 6378 km.

2. It takes 5.0 s after seeing a bolt of lightning to hear the thunder. If the speed of sound is 340 m/s, find the distance of the bolt of lightning from the observer. Assume no delay in light.

3. The highest frequency sound wave a human can hear is 20 kHz. Find the corresponding wavelength. Calculate the wavelength of the highest pitch sound audible by a dog who can hear as high as 45 kHz.

4. Calculate how far away a horn can be heard if the lowest audible intensity of a sound for a human is 1.0×10^{-12} W/m^2 and the power of the horn sound source is 2.0×10^{-5} W.

5. To determine if there is water in a well and the depth of the water, we drop a stone in the well. If we hear the sound after 3.0 s, calculate the depth of the well. Assume the speed of sound is 330 m/s.

6. Find the speed of sound if the frequency is 200 kHz and the wavelength is 0.5 cm.

7. A musical note produces a frequency of 400 Hz and has an intensity of $1.0 \times 10^{-6}\ W/m^2$. Find the amplitude of this sound wave.

8. Find the speed of sound in steel. Look up the necessary parameters on the web.

9. Find the speed of sound in water. Again look up the necessary parameters on the web.

10. In the previous problem, a submarine is pinging and there are three sonar buoys in a triangular array with each side 100 m on the surface of the water. If two buoys answer after 0.5 s while the third answers after 0.7 s, locate the position of the submarine. The speed of sound in sea water is 1481 m/s.

11. Calculate the intensity of a sound wave in the air if its amplitude is 1.0 cm and the frequency of the sound is 300 Hz.

12. The speed of sound in the air is 330 m/s. Calculate the fundamental frequency of an open-ended air column with a length of 90 cm.

13. If an ambulance is driving toward you at 100 km/h and sounds its siren at 10 kHz, calculate the frequency of the sound you hear. Assume the speed of sound is 340 m/s.

14. A man is standing 5.0 m from a railroad track, and hears the train traveling 45.0 m down the track moving toward the station at 30 km/h. Calculate the frequency of the sound the man hears if the frequency of the sound is 8000 Hz.

15. A person hears the clap of thunder 20 s after he/she sees the lightning. If the speed of sound in the air is 330 m/s and the speed of light is 300000 km/s, how far is he/she from the location of the lightning?

16. A steel wire has a tension of 100 N and has a linear density of 50.0 g/m. If the linear density of the wire under this tension decreases by about 0.001%, find the speed of the transverse wave in the wire.

17. A policeman is using radar to determine the speed of an oncoming car. If his radar gun has a frequency of 0.1 m and the reflected frequency of the radar changes by 10^{-5}%, find the speed of the car. Assume the speed of the radar wave is 300000 km/s.

Chapter 15

Heat and Thermodynamics

In this chapter we study the topic of *Heat and Thermodynamics*. This topic is an extension of work and energy studies we have done so far. In chapter 7, we stressed, for conservation of energy, that there are no frictional forces involved. This was done because friction creates heat that we cannot measure and therefore the total energy of the system cannot be accounted for. Before we delve into this very important topic in physics, we have to define some thermodynamics parameters and get familiar with the terminology and the *lingo*, so to speak, of the science of *Heat and Thermodynamics*.

15.1 Temperature

So far in our studies of mechanics, we expressed all mechanical quantities such as velocity, force, work, momentum, etc in terms of three indefinables, *length, mass, and time*. Now in the physics of *Heat and Thermodynamics*, where the non-mechanical aspects are obvious, we need a fourth quantity called the *Temperature*.

Temperature is a scalar quantity and it is a measure of the *hotness* or the *coldness* of an object. It is usually denoted with the lower case Latin letter t (not to be confused with time). It is common to confuse *Temperature* with *Heat*. The two although related, are not the same because as we shall see below *Heat* is a form of energy.

15.2 Units of Temperature

The unit of *Temperature* in the SI system is the *Celsius* ($°C$). The Celsius scale is calibrated such that at standard pressure, the temperature for freezing water is 0 $°C$ and for boiling water is 100 $°C$.

Degree Kelvin ($°K$) is simply a shift in 0 of the degree Celsius. Absolute zero is $-273.15\ °C$ or $-459.67\ °F$ and it is implied to be the lowest temperature attainable. The temperature of liquid Helium is measured to be $-269\ °C$ ($-452.2\ °F$), about 4 $°K$.

In the United States as well as Belize, Bermuda, Jamaica, Palau and the United States territories of Puerto Rico, Guam and the U.S. Virgin Islands the

Celsius unit is not used. Instead, the *Fahrenheit* scale is used to express official weather parameters and forecasts.

The unit of *Temperature* in the Fahrenheit scale is calibrated such that at standard pressure, the temperature for freezing water is 32 $^\circ F$ and for boiling water is 212 $^\circ F$.

The conversion equation between Celsius and Fahrenheit can be derived in the following manner. We note that there is a zero offset of (32 $^\circ F$) as compared to degree $^\circ C$, hence we can write;

$$^\circ C =^\circ F - 32. \tag{15.1}$$

The scale for Celsius is 100 and that of the Fahrenheit scale is $212-32 = 180$, we therefore have;

$$\frac{^\circ C}{100} = \frac{^\circ F - 32}{180}, \tag{15.2}$$

or,

$$\frac{^\circ C}{5} = \frac{^\circ F - 32}{9}. \tag{15.3}$$

Hence we can write;

$$\boxed{^\circ C = \frac{5^\circ F}{9} - \frac{160}{9}}. \tag{15.4}$$

Note according to equation 15.4 -40 ($^\circ C$) is the same as -40 ($^\circ F$).

Example 1. Calculate the equivalent temperatures in degree Celsius for (a) 5000 $^\circ F$, (b) 104 $^\circ F$ and (c) 450 $^\circ F$.

Answer:

(a) Using equation 15.3 at 5400 $^\circ F$, we have;

$$^\circ C = \frac{5^\circ 5400}{9} - \frac{160}{9}, \tag{15.5}$$

or,

$$\boxed{t \approx 2982^\circ C}. \tag{15.6}$$

(b) Using equation 15.3 at 104 $^\circ F$, we have;

$$^\circ C = \frac{5^\circ 104}{9} - \frac{160}{9}, \tag{15.7}$$

or,

$$\boxed{t = 40°C}. \tag{15.8}$$

(c) Using equation 15.3 at 450 $°F$, we have;

$$°C = \frac{5°450}{9} - \frac{160}{9}, \tag{15.9}$$

or,

$$\boxed{t \approx 232.2°C.} \tag{15.10}$$

As we observe from the above example, at high temperatures the quantity $\frac{160}{9}$ has minimal effect in the estimation of the temperature in degree Celsius. One can just multiply the temperature in degree Fahrenheit by ≈ 0.55 and the answer is fairly close to the actual value.

15.3 The Zeroth Law of Thermodynamics

Before we define the *The Zeroth Law of Thermodynamics* we must understand the "equilibrium" of a thermodynamical system.

A thermodynamical system composed of a liquid or a solid which is isolated from its surroundings approaches a state of thermal equilibrium if it is left undisturbed for a long period of time. Thermal equilibrium of a system is characterized by a set of physical quantities describing its state such as density, temperature, pressure, and volume which are independent of time.

If two systems are in contact then they reach thermal equilibrium. With the above knowledge we define *The Zeroth Law of Thermodynamics*.

The Zeroth Law of Thermodynamics states that if two systems A and B are in thermal equilibrium with system C, then systems A and B are also in thermal equilibrium with one another.

This fundamental law of thermodynamics is equivalent to one of Euclid's Axioms which states:

Things that are equal to the same thing are also equal to one another which is known as transitive property of equality.

All these laws and axioms are really very intuitive and common sense dictates their validity.

15.4 Thermal Expansion

The thermal expansion due to the thermal effects is one of the most important factors in engineering design. Many of us have experienced in our everyday lives that if we heat up an object, its length would increase.

An example of everyday experience with thermal expansion is in our kitchen. Almost all of us have wrestled with a stubborn metal lid on a glass jar. We probably learned from our grandmother that we should hold the lid under hot water and this makes it easier to open the lid. The reason is that metal expands more quickly than glass and we can open the jar easily.

Mathematically, the expansion of materials can be understood as an increase in temperature ΔT which causes an increase in inter-atomic dimensions which then manifests itself in a macroscopic expansion of the object. If we denote the expansion in the length of an aluminum bar Δl, then we can write;

$$\boxed{\Delta l = \alpha l \Delta T}. \tag{15.11}$$

In equation 15.11 l is the length at the initial temperature T_i. α is the *Coefficient of Linear Expansion*, and strictly speaking, has some dependency on the temperature but it is negligible and can be neglected for engineering design purposes.

Example 2. A Vernier Caliper 15.0 cm long is designed to measure to a tenth of a millimeter or 10^{-4} m. If the *Coefficient of Linear Expansion*, α, for stainless steel is $1.73 \times 10^{-5}/°C$ calculate the maximum allowable temperature variation to maintain the accuracy of 10%.

Answer:

We have to find the elongation or the contraction of the vernier for a 10% variation and that is our allowable Δl.

$$\Delta l = 10^{-4} \times 0.1 = 10^{-5}. \qquad (15.12)$$

Now that we have Δl, we proceed with equation 15.11, and rearranging it;

$$\Delta T = \frac{\Delta l}{\alpha l}. \qquad (15.13)$$

Plugging the known values into 15.13 we have;

$$\Delta T = \frac{10^{-4}}{1.73 \times 10^{-5} \times 0.15}, \qquad (15.14)$$

or;

$$\boxed{\Delta T \approx 38.54°C}. \qquad (15.15)$$

We can show that for area and volume expansion the area and the volume increase due to thermal expansion, given the fact most solids expand isotropically, are approximately;

$$\boxed{\Delta A \approx 2\alpha A \Delta T}, \qquad (15.16)$$

and for the volume;

$$\boxed{\Delta V \approx 3\alpha V \Delta T}. \qquad (15.17)$$

15.5 Heat

As eluded to in the beginning of this chapter, Heat is a form of energy. Heat can be generated by mechanical energy and it also can generate mechanical energy. When you rub your hand against a rough fabric, the faster you rub, the hotter it gets. This is because friction force between your hand and the fabric creates heat. We therefore convert mechanical energy into heat. The inverse of this process is also possible by, for example, boiling water and making steam which can be used to drive a turbine and therefore create mechanical energy.

The concept of heat being a form of energy was not clear in the 19^{th} century. It was in 1851, that William Thomson in his "On the Dynamical Theory of Heat", proposed the idea. Based on experiments by James Joule, he concluded that heat is not a substance but a dynamical form of mechanical effect. Hence we perceive that there must be an equivalence between mechanical work and heat, as there is one between cause and effect.

There obviously must be a relation between the quantity of heat and a change in temperature. This relation is a linear relation and it is through a constant called the *heat capacity*. Hence we can write;

$$C = \frac{\Delta Q}{\Delta T}. \tag{15.18}$$

Equation 15.18 expresses mathematically the rise in heat added per unit temperature increase.

Now we define another constant as the ratio of the *heat capacity* to the mass of a substance called the *specific heat* which then is a property of a given substance. We therefore can write;

$$\boxed{c = \frac{\Delta Q}{m\Delta T}}. \tag{15.19}$$

The unit of *specific heat* in the *SI* system is $J/kg°K$. Note *specific heat* like many constants in physics is not a true "constant" and it changes with temperature and pressure. For example, the specific heat of water varies by 1% if the temperature increases from 0 $°C$ to 100 $°C$.

We can also rewrite equation 15.19 in terms of the amount of heat absorbed by a substance.

$$\boxed{\Delta Q = mc\Delta T}. \tag{15.20}$$

Note the temperature is in degree Kelvin. Although the difference in temperature does not make any difference if we subtract two temperatures in degree Kelvin or Celsius, we should be aware of the actual unit of temperature in equation 15.20.

The unit of heat is the same as energy and it is measured in *Joule* in the SI system. Another very popular unit for heat is the *calorie*. Each calorie is equal to 4.184 J.

15.6 Heat Conduction

Heat transfer occurs from a high temperature region to a low temperature region. The transfer of heat or energy from a high temperature to a low temperature region of a body is called *Heat Conduction*. Let us study a slab of material with cross sectional area A and thickness Δx where the two faces are kept at different temperatures. From this study, we note that the rate of change of heat with time is equal to the gradient of temperature along the thickness of the slab, Δx. We therefore can write the *Fundamental Law of Heat Conduction* as;

$$\boxed{\frac{dQ}{dt} = kA\frac{dT}{dx}}. \tag{15.21}$$

The constant k is called the *Thermal Conductivity* and it is large for metals and small for gases, nonmetals and insulators. For steady state processes, the temperature at each point along all cross sections is a constant and we can then write;

$$\boxed{\frac{dQ}{dt} = kA\frac{T_2 - T_1}{L}}. \tag{15.22}$$

15.7 Heat and Work

The mechanical work we defined in chapter seven is;

$$W = \int_{x_1}^{x_2} \mathbf{F}.\mathbf{dx}. \tag{15.23}$$

However, we know that $F = pA$ and we therefore can substitute for F in equation 15.23 and noting that $dV = Adx$, we have;

$$W = \int_{V_1}^{V_2} pdV. \tag{15.24}$$

15.8 First Law of Thermodynamics

Before we delve into the definition of *First Law of Thermodynamics* we define a thermodynamic quantity known as the *Internal Energy.* The internal energy denoted in the literature by the capital Latin letter U is defined as follows.

Definition: *The internal energy is the energy stored in the system at a microscopic level due to the vibration of the atoms in a substance plus the binding energy which holds the atoms together.*

We therefore can realize the internal energy U as the microscopic energy contained within a system

Now, we can define *First Law of Thermodynamics* as follows.

Definition: *For a system in mechanical equilibrium, any change in the internal energy of a system* ΔU *is either because of the heat flow into or out of the system or because of the work done by or on the system.*

We can describe the *First Law of Thermodynamics* mathematically as;

$$\boxed{\Delta U = Q + W}. \tag{15.25}$$

In equation 15.25 we have used the International Union of Pure and Applied Chemistry (IUPAC) nomenclature, where W is assumed to be the work done on the system. It is important to note that the system is in static equilibrium and that the center of mass of the system is at rest.

15.9 Problems

1. At what temperature are the readings at Celsius and Fahrenheit the same? At what temperature would the Fahrenheit scale read twice that of the Celsius scale?

2. The approximation of $°F = 1.8\ °C$ would give a good mental math solution for conversion of Fahrenheit to the Celsius scale. At what temperature would this approximation introduce an approximately 3.2% error?

3. A brass rod 50 cm long is subjected to a heat source and as a result its temperature rises by 25 $°C$. Calculate the increase in the length of the rod.

4. Railroad engineers design the rails with periodic gaps to prevent bending of the rails due to increase in temperature in the summer season and subsequent elongations of rails. If the coefficient of linear expansion for Carbon Steel is $10.8 \times 10^{-6}\ K^{-1}$, and if the length of rail between two consecutive gaps is 100 m and the gap is 25 cm and the temperature rise from the winter to the summer is 25 $°C$, then is the gap wide enough to avoid bending of the rails? Quantify your answer!

5. We mentioned without proof that the approximate equation for surface expansion is $\Delta A \approx 2\alpha A \Delta T$. Prove this approximate equation and state what approximation you made.

6. Repeat problem 5 for the approximate volume expansion.

7. In a sensor, a radioactive source is to move with constant velocity on a radiation detector. We achieve this by mounting the radioactive source to the end of an aluminum rod about 5 cm long and heat up the rod with a heat source. At what constant rate should we change the temperature of the rod to move the radioactive source at a speed of 1 $\mu m/s$?

8. The diameter of a hole in a steel plate at 300 $°K$ is 3.0 cm. Calculate the diameter of the hole if the temperature rises to 200 $°C$.

9. A steel rod is mounted between two rigid walls and the entire structure is at a room temperature of 300 $°K$. If the rod is 5 cm in diameter, calculate the stress in the rod if the temperature is raised by 30 $°C$.

10. An aluminum bar has a diameter of 2.0 cm. What is the minimum force that prevents it from elongating if its temperature increases from 300 $°K$, to 100 $°C$?

11. A sphere made of brass has a diameter of 4.0 cm and has a density of 8.4 g/cm^3. What is the percentage increase in the density if the temperature is dropped by 100 $°K$?

12. Prove that the change in density of a substance is approximately;

$$\Delta\rho = -\gamma\rho\Delta T. \tag{15.26}$$

Explain the minus sign. Cite an example where the minus sign does not apply.

13. A grandfather clock has a pendulum which has a period of 1.0 s. The pendulum is made of brass and the temperature of the room is kept at 25 $^\circ C$ to keep the display of time accurate to a minute per month. If Grandpa is out of town for a month and the AC breaks down as soon as he leaves and the temperature rises to 330 $^\circ K$, how much would his clock be off when he gets back?

14. Composite materials are widely used in industry because of their strength and their light weight. However, the thermal stresses due to different coefficients of thermal expansion for various components of the composites are of great engineering design concern. If we reinforce a ceramic called *zirconia porcelain* with a Young Modulus of 170 GPa and a coefficient of linear expansion $\alpha = 4.5 \times 10^{-6}$, with 1.0 cm diameter steel rods, find the maximum allowable temperature change which would not create cracks in the ceramic.

15. A high-voltage power line made of copper is stretched tight and straight between two poles 50 m apart in the winter time at -10 $^\circ C$. Calculate how far the power line sags in the summer time when the temperature rises to 30 $^\circ C$.

16. Newton's Law of Cooling is defined as a linear proportionality between the rate of change of the temperature of an object and its temperature, within limited temperature variations. Mathematically we can write;

$$\frac{dT}{dt} = -KT. \tag{15.27}$$

The minus sign signifies that the object is cooling down with increasing time. Find the solution to the above differential equation.

17. A high-voltage power line made of copper is stretched between two poles 50 m apart. If the diameter of the wire is 1.0 cm and the line undergoes a 20 $^\circ C$ increase in temperature due to a sudden jump in the electrical current, find the amount of heat generated by the cable. Look up all necessary parameters on the web.

18. A hot carbon steel bar at 600 $^\circ C$ is 20.0 cm long and has a diameter of 1.0 cm. The bar is then quenched in a 50 l ice-water bath initially at 0 $^\circ C$. If the bar is quickly removed from the water bath and its temperature drops to 100 $^\circ C$, find the increase in the water temperature. Look up all necessary parameters on the web.

19. A car weighing 9800 N traveling at 20.0 m/s comes to rest by applying its brakes. Calculate the amount of heat generated by the brakes.

20. A $100 - g$ projectile traveling at 300 m/s hits a $10.0 - kg$ sand bag and comes to rest. Assuming a stationary bag, calculate the amount of heat generated by the projectile. Look up all necessary parameters on the web.

21. A $5.0 - kg$ block of ice at a temperature of 273 $^\circ K$ slides down a rough 30° inclined surface from a height of 20 m and when it reaches the bottom, its mass reduces by 10%. Calculate the temperature of the ice at the bottom of the inclined plane.

22. My caloric intake per day is 2200 $kCal$. If I expend all these calories in the form of heat, how does this compare to a $60 - W$ light bulb?

23. If heat is being transferred through a 20 cm steel rod at a rate of 5 calories per minute, calculate the temperature difference of the two ends of the steel rod after 1 minute. Calculate the specific heat of 1.0 kg piece of metal at a temperature of 400 $^\circ C$ submerged in water at an initial temperature of 300 $^\circ K$, if the final temperature of the metal is 20 $^\circ C$.

24. Find the total heat gained by 10.0 kg of water heating up from 20 $°C$ to 373 $°K$.

25. A steel ball of mass 500 g is dropped from a height of 10.0 m and hits the ground at a velocity of 13.8 m/s. Find the total heat generated due to air resistance in calories.

26. Calculate the amount of heat required to melt 1.0 kg of ice at a temperature of -20 $°C$ to steam at 100 $°C$.

27. 2.0 kg of ice at a temperature of 260 $°C$ is dropped into a steel container of water at a temperature of 50 $°C$. The steel container weighs 9.8 N. Assuming no heat loss, calculate the final temperature of the system.

28. Calculate the change in internal energy if the applied work is 3000 J and the amount of heat is 2000 J.

29. A concrete slab is 20 cm thick and has a thermal conductivity of 0.70 $W/°C$. If the temperature on one face of the slab is 260 $°K$ and on the other side 75 $°F$, find the heat transfer per unit area.

30. A $5.0 - kg$ block of ice at -5 $°C$ is dropped in a water container initially at 15 $°C$. If the final temperature of the system is 3 $°C$, find the mass of the water.

31. A bolt with a diameter of d fits perfectly in a hole with the same diameter both at temperature T. If the bolt temperature increases by ΔT and as a result its diameter increases to D, find the mass of the water bath at t $(t < T)$ required to reduce the diameter of the bolt to d.

Chapter 16

The Kinetic Theory of Gases

In chapter 15, we studied macroscopic quantities such as *temperature*, *pressure* and *heat* without delving into the cause of these *thermodynamical* quantities at a molecular or atomic level. The topics in this chapter are extensions of studies we have done so far but at a microscopic level. In chapter 7, in discussing conservation of energy, we emphasized that there are frictional forces involved. This was done because friction creates heat that we cannot measure and therefore the total energy of the system cannot be determined. However, in reality, every physical process undergoing a change in temperature or pressure creates kinetic energy and vice versa. The increase in the kinetic energy of the atoms or molecules in a gas is what causes the increase in pressure due to the collision of these atoms or molecules with a container wall, for example. It is rather obvious to say that there is no way one can follow the dynamics of each and every atom in a given gas. It is however possible to derive macroscopic quantities such as *pressure* statistically and derive them as average quantities. This branch of *thermodynamics* is referred to as *statistical mechanics*. The task of following the kinetic energy of each molecule or atom is beyond the capability of any computer! Before we delve into this very important topic in physics, we need to define physical quantities related to *statistical mechanics*.

16.1 Mole and the Avogadro's Number

In Physics and Chemistry the *mole* is a unit of measurement which expresses the amount of a given substance that contains the same number of atoms or molecules as 12 g of $carbon-12$ or ^{12}C. $Mole$ is a SI unit and it is denoted by the symbol mol.

The number of atoms in 12 g of ^{12}C is called the *Avogadro's Number or Constant* and is $6.02214129(27) \times 10^{23}$. The 12 g of carbon is called the *Atomic Weight* of carbon. The *Avogadro's Number* is defined in such a way that the mean atomic or molecular weight in grams of any substance or chemical compound contains exactly $6.02214129(27) \times 10^{23}$ atoms or molecules. For example, the mean atomic weight of natural copper (Cu) is 63.546 g/mol. This means that 63.546 g of natural copper contains $6.02214129(27) \times 10^{23}$ copper atoms.

Measurements of *Avogadro's Number* have improved significantly since *Amadeo Avogadro* proposed in 1811 that the volume of any gas at a certain pressure is proportional to the number of atoms or molecules in the gas. Today's measurement of *Avogadro's Number* is performed and refined by the International Avogadro Coordination (IAC) often called the "Avogadro project". The goal of IAC which is an international collaboration founded in the early 1990s composed of various national metallurgy institutes is to measure the Avogadro constant. These measurements are performed by the X-ray crystal density technique to a relative uncertainty of 2×10^{-8}.

Example 1. The IceCube Neutrino Observatory located at the geographic South Pole is a cubic kilometer of instrumented ice consisting of 5160 photomultiplier tubes mounted on 86 strings deployed in a hexagonal array at a depth of $1.5 - 2.5\ km$. The primary goal of the IceCube experiment was the detection of astrophysical neutrinos which has now been realized (Science, November 22, 2013). In order to calculate the number of neutrino interactions in the IceCube Detector we must first find the number of water molecules in one cubic kilometer of ice. If the density of ice is $0.917\ g/cm^3$ and one mole of ice is $18.0152\ g$, calculate the number of water molecules in the IceCube detector.

Answer:

First we must calculate the number of moles of H_2O in a km^3 of ice. Remember $1.0\ km = 100000\ cm$.

$$M_{ice} = \rho_{ice} \times V_{ice}, \tag{16.1}$$

or;

$$M_{ice} = 0.917 \times (100000)^3 = 9.17 \times 10^{14}\ g. \tag{16.2}$$

To find the number of moles, we must divide the mass M_{ice} by the molecular weight of water which is $18.0152\ g$.

$$Number\ of\ moles = \frac{9.17 \times 10^{14}}{18.0152} \approx 5.1 \times 10^{13} \tag{16.3}$$

Now if we multiply the number of moles obtained in equation 16.3 by *Avogadro's Number*, we then obtain the desired result and determine the number of water molecules in the IceCube detector.

$$\boxed{\#\ of\ ice\ molecules = 5.1 \times 10^{13} \times 6.02214129 \times 10^{23} \approx 3.7 \times 10^{37}} \tag{16.4}$$

16.2 Ideal Gas

In order to bridge the gap between the microscopic world of atoms and molecules and the macroscopic world of thermodynamic quantities such as pressure and temperature, we need to define and simplify the physical model of gases so that we can describe them mathematically.

Let us assume we have a container filled with a gas with a mass M at a pressure p and a temperature T. If we remove some of the gas, then the pressure will drop, or if we increase the volume the pressure will drop, or finally if we decrease the temperature, then again the pressure will drop. This simple experiment which seems intuitive tells us that pressure and volume of a gas are inversely proportional while pressure and temperature are directly proportional. We can express this relation in the following mathematical form.

$$pV = CT \tag{16.5}$$

The above equation is the essence of the *Ideal Gas*. We therefore can define an ideal gas as;

Definition: *a collection of molecules or atoms of gas where only elastic collisions are possible and that the interactions among atoms and molecules are negligible.*

In equation 16.5, the constant C can be written as the product of two quantities namely the molar mass of the gas n and a constant R called the *Universal Gas Constant* and it has a value of 8.3145 $J/mol - K$. Hence we can write the *Ideal Gas Law* as;

$$\boxed{pV = nRT}. \tag{16.6}$$

It is instructive to find the volume of a mole at *Standard Pressure and Temperature* (STP), that is at 273 $^\circ K$ and one atmosphere of pressure at 101.325 kPa according to the National Institute of Standards and Technology (NIST).

$$V = \frac{nRT}{p} = \frac{1\ mol \times 8.3145\ J/mol - K \times 273}{101.325} \approx 22.4\ l \tag{16.7}$$

16.3 Temperature and Pressure of Gases

The ideal gas law of equation 16.6 provides a tool by which we can study thermodynamical processes in many gases. We see from the ideal gas law that for a fixed number of moles n given the fact that R is constant, then $p,\ V,\ and\ T$ are the only variables. This observation provides a tool to study and calculate properties of gases as these variables change. This is best illustrated through examples.

Example 2. A cylinder of volume $V = 2.0$ liters at a pressure of 10 atm is at a room temperature of 25 $^\circ C$. The temperature is increased by 10 $^\circ C$, calculate the pressure of the gas in the cylinder.

Answer:

The ideal gas law for the initial and the final temperatures is;

$$p_i V = nRT_i ; p_f V = nRT_f . \tag{16.8}$$

Furthermore, we can write this equation for the two temperatures, after we convert degree Celsius to degree Kelvin.

$$T_i = 273 + 25 = 298\ ^\circ K;\ T_f = 273 + 35 = 308\ ^\circ K \tag{16.9}$$

We should note that, although the volume is specified as 10 l, it is redundant information and can be ignored because it remains constant. We therefore can write;

$$\frac{p_i}{T_i} = \frac{p_f}{T_f}. \tag{16.10}$$

We are searching for the final pressure, p_f, then we have;

$$p_f = \frac{p_i T_f}{T_i}. \tag{16.11}$$

Substituting the known values for p_i, T_i, and T_f, we get;

$$\boxed{p_f = \frac{10 \times 308}{298} \approx 1.035\ atm} \tag{16.12}$$

As we studied in chapter 15, the expanding gases can do work and we can write;

$$W = \int_{V_1}^{V_2} pdV. \tag{16.13}$$

In equation 16.13 we can substitute for p, its value from the ideal gas law which is $\frac{nRT}{V}$. We therefore can write;

$$W = \int_{V_1}^{V_2} \frac{nRT}{V} dV. \tag{16.14}$$

Assuming that the temperature of the process is constant, we can then write;

$$W = nRT \int_{V_1}^{V_2} \frac{dV}{V}. \tag{16.15}$$

Integrating equation 16.15 and plugging in the limits on the volumes, we obtain the work done.

$$W = nRT \ln \frac{V_2}{V_1}. \tag{16.16}$$

Figure 16.1 shows the process of a constant temperature discussed above. This type of process is called an *isothermal process*. The work done by the piston is the area under the curve in the lower plot of figure 16.1. We can also think of the gas expanding from the top figure b) and ending up in the state of top figure a) indicating that the gas works on the piston.

16.4 Kinetic Energy of Gases

The motion of every gas molecule or atom, in the case of *noble* gases, carry kinetic energy. This kinetic energy is proportional to either the *pressure* or the *temperature* of the gas. We derive these relations in the subsection below.

16.4.1 Kinetic Energy and Pressure

Let us assume gas molecules are in a cubic container of side l and let us look at the momentum of a single molecule as it collides with the walls perpendicular to the $x - direction$. The x component of the momentum of the molecule is then;

$$\Delta p_x = p_{xf} - p_{xi} = m(v_x + -(-v_x) = 2mv_x. \tag{16.17}$$

Note, the final momentum and the initial momentum have the same magnitude but opposite directions hence requiring the minus sign in equation 16.17. We also know that;

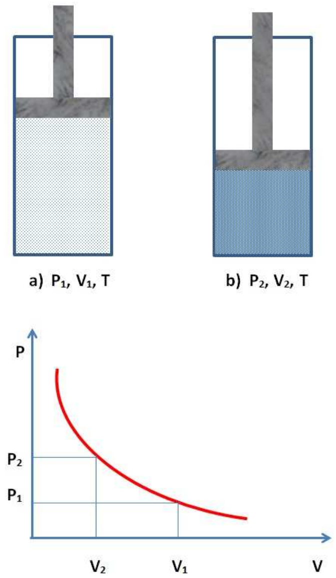

Figure 16.1: A volume-adjustable gas-filled cylinder showing the relation among pressure, volume and temperature. The plot in the lower part of the figure clearly indicates the inverse relation between the pressure and the volume of the gas. Note, the temperature remains constant through the process referred to as *isothermal process*.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{16.18}$$

We also recall that work is simply the dot product of the force and the distance. Therefore, the above molecule can travel a distance $2l$ (one side of the box and back) and the work done will be;

$$W = (\frac{\Delta 2p_x}{\Delta t})(2l). \tag{16.19}$$

In equation 16.19, v_x is equal to $\frac{2l}{\Delta t}$, we can then rewrite 16.19 as;

$$W = mv_x^2. \tag{16.20}$$

Therefore, the total work done is the sum of the work done in all three directions;

$$W = m\{v_x^2 + v_y^2 + v_z^2\}. \tag{16.21}$$

Now we sum over N molecules in the gas;

$$W = \sum_{i=1}^{N} m_i(v_{ix}^2 + v_{iy}^2 + v_{iz}^2). \tag{16.22}$$

Now we can assume that the average velocity is $\bar{v}^2/3 = v_x^2 + v_y^2 + v_z^2$ and since all molecules have the same mass, then equation 16.22 becomes;

$$W = Nm\bar{v}^2/3. \tag{16.23}$$

From equation 16.13 we can write;

$$W = pV = Nm\bar{v}^2/3. \tag{16.24}$$

Hence the pressure becomes;

$$p = \frac{Nm\bar{v}^2}{3V}. \tag{16.25}$$

Note, the kinetic energy is $T = mv^2/2$ and we can rewrite equation 16.25 as;

$$\boxed{p = \frac{2T}{3V}}. \tag{16.26}$$

Therefore, the kinetic energy is;

$$\boxed{K = \frac{3}{2}pV}. \tag{16.27}$$

Equation 16.27 is a relation between the *microscopic* variable, the kinetic energy to the *macroscopic* variable pressure.

16.4.2 Kinetic Energy and Temperature

From the *ideal gas law* $pV = nRT$, we can replace the right side of equation 16.27 with the right side of the *ideal gas law* and then we can write;

$$\boxed{K = \frac{3}{2}nRT}. \tag{16.28}$$

Now we replace the constants n and R with their values and we then define the constant k as the *Boltzmann Constant*. Hence, we have;

$$k = \frac{8.3145}{6.022 \times 10^23} = 1.38065 \times 10^{-23}. \tag{16.29}$$

We can then write the kinetic energy in terms of the temperature of the gas;

$$\boxed{K = \frac{3}{2}kT}. \tag{16.30}$$

The kinetic energy for one mole of an ideal gas is therefore;

$$\boxed{K = \frac{3}{2}RT}. \tag{16.31}$$

Example 3. Calculate the total kinetic energy of 1 mol of an ideal gas at room temperature $t = 25\ ^\circ C$. Assuming the gas is Neon with an atomic mass of 20.1797 g, calculate the average velocity of a typical Ne atom in the gas.

Answer:

We use 16.31 and converting the 25 $^\circ C$ to $^\circ K$;

$$T = t + 25 = 273 + 25 = 298\ ^\circ K, \tag{16.32}$$

and then;

$$\boxed{K = \frac{3}{2}8.3145 \times 298 \approx 3717\ J}. \tag{16.33}$$

The atomic mass of Ne is 20.1797 g/mol which is 0.0201797 kg/mol.

$$K = 3717\ (J) = \frac{1}{2} 0.0201797\ (kg) \bar{v}^2. \tag{16.34}$$

Then;

$$\boxed{v \approx 607\ m/s}. \tag{16.35}$$

It is interesting to note that the above result is approximately twice the speed of sound in air!

16.4.3 Specific Heat of Gases

In the previous sections, we defined the ideal gas to be a collection of atoms or molecules approximated as hard spheres. We also assumed that the inter-atomic or inter-molecular interactions were negligible. These assumptions greatly simplify the situation and we can conclude that the internal energy of ideal gases is simply equal to their kinetic energy. In the previous sub-section we derived the average value for the kinetic energy per molecule or per atom as $\frac{3}{2}kT$. We therefore can conclude that for an ideal gas the internal energy is simply proportional to its *temperature*. Hence, we can write the heat gained or lost by a thermodynamical system from the first law of thermodynamics as;

$$\Delta Q = \Delta U + \Delta W. \tag{16.36}$$

We learned in the previous chapter that the *specific heat* is defined as the heat required per unit mass per unit temperature increase. If we adopt the mass unit to be mole, then the corresponding specific heat is referred to as *molar specific heat*. There are two *specific heats* for gases, *Constant Volume Specific Heat;* C_v and *Constant Pressure Specific Heat;* C_p. In previous sub-sections we learned that the work done is $\Delta W = p\Delta V$. If we assume there is no change in volume then $\Delta W = 0$. We then can write equation 16.36 as;

$$\Delta U = nC_v\Delta T. \tag{16.37}$$

In equation 16.37, n is the number of moles. For n moles of a gas the kinetic energy and therefore the internal energy is $\Delta U = \frac{3}{2}nR\Delta T$. Now we can write equation 16.37 as;

$$nC_v\Delta T = \frac{3}{2}nR\Delta T, \tag{16.38}$$

or;

$$\boxed{C_v = \frac{3}{2}R}. \tag{16.39}$$

Conversely, we study the case when pressure is constant but both volume and temperature change. Then the work done by the system is not zero and we can write the first law as;

$$nC_p\Delta T = nC_v\Delta T + p\Delta V. \tag{16.40}$$

However, we know that;

$$p\Delta V = nR\Delta T. \tag{16.41}$$

Combining the two equations 16.40 and 16.41 we obtain the following relation between C_v, C_p and R;

$$C_p - C_v = R \tag{16.42}$$

From the two equations 16.39 and 16.42 we find that

$$\boxed{C_p = \frac{5}{2}R}. \tag{16.43}$$

16.5 Adiabatic Processes

Adiabatic processes in thermodynamics are referred to as thermodynamical processes wherein there is no heat flow in or out of a system, i.e., no transfer of heat. We can mathematically derive this concept from the first law of thermodynamics.

If there is no change in heat, then $\Delta Q = 0$ and the first law then becomes;

$$nC_v\Delta T + p\Delta V = 0, \tag{16.44}$$

or;

$$\Delta T = -\frac{p\Delta V}{nC_v}. \tag{16.45}$$

However;

$$\Delta U = \Delta pV = p\Delta V + V\Delta p. \tag{16.46}$$

Equation 16.46 can also be written via the ideal gas law as;

$$p\Delta V + V\Delta p = nR\Delta T. \tag{16.47}$$

Therefore we can write;

$$\Delta T = \frac{p\Delta V + V\Delta p}{nR}. \tag{16.48}$$

If we equate the right-hand sides of the two equations 16.45 and 16.48, we get;

$$-\frac{p\Delta V}{nC_v} = \frac{p\Delta V + V\Delta p}{nR}. \tag{16.49}$$

Recall that $C_p - C_v = R$ and simplifying equation 16.49 we then have;

$$C_v p\Delta V + C_v V\Delta p = -Rp\Delta V = C_v p\Delta V - C_p p\Delta V. \tag{16.50}$$

Further simplification of 16.50 yields;

$$C_p p\Delta V + C_v V\Delta p = 0. \tag{16.51}$$

Dividing equation 16.51 by $C_v pV$ and assuming differential changes in volume and pressure we have;

$$\frac{dp}{p} + \frac{C_p}{C_v}\frac{dV}{V}. \tag{16.52}$$

We denote $\gamma = \frac{C_p}{C_v}$ and assume it is constant, then we can write

$$\boxed{\frac{dp}{p} + \gamma\frac{dV}{V}}, \tag{16.53}$$

integrating equation 16.53 we have;

$$\boxed{\ln p + \gamma \ln V = Constant}. \tag{16.54}$$

Some algebra will further simplify equation 16.54;

$$\ln p + \ln V^{\gamma} = Constant, \tag{16.55}$$

or,

$$\ln pV^{\gamma} = Constant, \tag{16.56}$$

and finally we can write;

$$\boxed{pV^{\gamma} = Constant}. \tag{16.57}$$

Note the magnitude of the "constant" depends on the amount of the ideal gas.

16.6 Problems

1. An adult human contains about 60% water by weight. Calculate the number of water molecules in an $80 - kg$ person.
2. Ten moles of an ideal gas undergoes an isothermal expansion of 10%. Calculate the heat absorbed by the gas.
3. A soccer ball has a pressure of 0.85 atm and a radius of 10.9 cm at 45 $°F$. If the temperature of the air in the ball increases to 20 $°C$ and the radius increases to 11.0 cm find the pressure of the ball.
4. Calculate the pressure of 100 g of Ne gas at a temperature of 100 $°C$
5. A sample of an ideal gas has a volume of 5.0 l at 25.0 $°C$ and 1.0 atm pressure. If we compress the gas so its volume decreases to 4.0 l and temperature 80 $°C$, what is the pressure of the gas?
6. The kinetic energy of a cold neutron is 0.020 eV. Find the temperature of the neutron.
7. If an electron has a temperature of -272 $°C$. Find its velocity at this temperature.
8. If the temperature of a sample of Xe gas changes from 40 $°C$ to 55 $°C$, what is the change in its velocity?
9. How much energy is stored in 50 mol of oxygen gas with a temperature of 50 $°C$?
10. What is the work done by a gas at 5.0 atm undergoing an isobaric expansion from 20 l to 40 l?
11. A bubble of 2.0 moles of N_2 gas is trapped under water. The water then heats up from 50 $°F$ to 75 $°F$. How much energy is absorbed by the N_2 gas?
12. In problem 11 find the amount of internal energy increase of the N_2 gas.
13. In problem 11, calculate the work done by the bubble.
14. The radius of Pluto is approximately 1680 km. If the atmospheric pressure is 0.3 Pa, find the number of moles of N_2 in the atmosphere. Assume the atmosphere of Pluto is predominantly N_2.
15. On a balmy day on Pluto, the temperature rises by 75 $°F$. Find the amount of internal energy increase of the N_2 gas in the atmosphere of Pluto.
16. A laboratory vacuum pump maintains a very low pressure of 10^{-8} atm. If the vacuum chamber is located in a room with a temperature of 25 $°C$, find the number of gas molecules in 1.0 cm^3.

Chapter 17

Entropy and Second Law of Thermodynamics

The *Entropy* is a measure of the *order or disorder* of a given system. This is the essence of the law of entropy or the second law of thermodynamics. The idea of the law of Entropy goes beyond mere thermodynamics and it has implications in biology, ecology and the overall aging of the universe. Any mechanical system, including thermodynamical ones, always strives to reach equilibrium. For example, if we increase the temperature of a system, we are in effect putting energy into the system. Now leaving the system alone, the temperature will decrease and the entropy increases and the system then reaches a state of equilibrium.

In contrast to the first law of thermodynamics which is the statement of conservation of energy and does not have a preferred direction, i.e. heat can do work and work can generate heat, the second law has only one direction.

The second law of thermodynamics states that the *entropy* of an isolated system always increases. This is because the system strives to reach *thermodynamic equilibrium* and therefore the state of *maximum entropy*.

17.1 Reversible and Irreversible Processes

In the 1850's, Rudolf Clausius was the first to mathematically treat *irreversibility* phenomena in nature through the concept of entropy.

Clausius states, in his 1854 memoir entitled; "On a Modified Form of the Second Fundamental Theorem in the Mechanical Theory of Heat":

"It may, moreover, happen that instead of a descending transmission of heat accompanying, in the one and the same process, the ascending transmission, another permanent change may occur which has the peculiarity of not being reversible without either becoming replaced by a new permanent change of a similar kind, or producing a descending transmission of heat."

For a reversible process, Clausius equality, stated mathematically, can be expressed as;

$$\oint \frac{dQ}{T} = 0. \tag{17.1}$$

The line integral of equation 17.1 is independent of the path. Now we define *Entropy* as;

$$dS = \frac{dQ}{T}. \tag{17.2}$$

Equation 17.2 states mathematically that the only possible way for the entropy to remain constant is for ds to vanish at absolute zero! This is known as the *Third Law of Thermodynamics*. The third law of thermodynamics refers to the properties of systems in equilibrium at absolute zero and it states:

Definition: *The entropy of a perfect crystal, defined as one that contains no point, linear, or planar imperfections, at absolute zero degree Kelvin, is exactly zero.*

This observation also tells us that for any other temperature above absolute zero there is an increase in entropy and the higher the temperature, the lower the increase in entropy for a given amount of heat transfer. Clausius inequality on the loop integral of equation 17.1 is,

$$dS \geq \int \frac{dQ}{T} \tag{17.3}$$

Note the equality only holds if the process is reversible. For adiabatic processes where $\delta Q = 0$, then $\Delta S \geq 0$.

The idea of reversible and irreversible processes in thermodynamics have their roots in mechanics. In mechanics, we described*conservative* forces as;

$$\mathbf{F} = -\nabla V. \tag{17.4}$$

In other words, a conservative force can be expressed as the *gradient*, of a potential, i.e., tangent to the curve of the potential. This requirement makes the work done by a conservative force independent of the path taken. For example, gravity in the absence of air resistance is a conservative force. On the other hand friction force is a nonconservative force because it depends on the path taken.

17.2 Enthalpy and Latent Heat

Latent heat or *Enthalpy* is the energy released or absorbed by a body during a thermodynamical isothermal process. The notion of Latent Heat was first introduced by Joseph Black (1762). The term is derived from Latin *latere* which means to be hidden. Latent heat examples are phase changes in matter, such as melting (fusion) or evaporation (boiling). The reverse processes are also possible, meaning condensation of liquid into solid and vapor into liquid.

The two most common forms of latent heat or enthalpies and their reverse processes are the two isothermal processes of melting or *fusion* and evaporation or *boiling*. In these two processes, heat is absorbed from a reservoir at constant temperature and therefore they are referred to as *endothermic*. In the opposite direction where the system releases heat to the outside to condense from liquid into solid or from gas into liquid, the process is called *exothermic*.

In thermodynamics, we usually deal with *specific latent heat* (L) which is referred to as the amount of heat per unit mass required to completely change the phase of a substance. We can therefore write;

$$L = \frac{Q}{m}. \tag{17.5}$$

The *specific latent heat* L is an intrinsic property of materials and tables are available for many substances on line. From equation 17.5, the latent heat, Q, for a substance with a mass m is;

$$\boxed{Q = mL}. \tag{17.6}$$

In equation 17.6 Q is the amount of heat released or absorbed, m is the mass of the substance and L is the *specific latent heat* for a given substance.

Example 1. How much latent heat is required to melt 5.0 kg of ice at 0 $^{\circ}C$ to water at 0 $^{\circ}C$?

Answer:

Using equation 17.6, we can write;

$$Q = 5.0L. \tag{17.7}$$

We look up the specific latent heat for ice and it is $3.3 \times 10^5\ J/kg$. Then;

$$Q = 5.03.3 \times 10^5, \tag{17.8}$$

or,

$$\boxed{Q = 1.65 \times 10^6\ J}. \tag{17.9}$$

17.3 Carnot Cycle

Carnot Cycle is a theoretical thermodynamical engine where a reversible process is possible and implies the limiting case of the *second law of thermodynamics* and involves no change in *entropy*.

The *Carnot Cycle* is comprised of four steps; two *isothermal* and two *adiabatic* processes. These processes are the work done by the engine through the expansion and the compression of the gas in the system and are listed below;

- A reversible isothermal expansion process where the temperature of the gas remains constant.
- A reversible adiabatic expansion process, where there is no heat transfer to the outside.
- A reversible isothermal compression process where the engine loses heat to the outside.
- A reversible adiabatic compression process where no heat transfer occurs.

The efficiency of the Carnot engine can be written as;

$$\eta = \frac{W}{Q}. \tag{17.10}$$

W is the work output and the Q is the input heat. Note the input Q is coming from a hot reservoir and the work done $W = Q_H - Q_C$, where Q_H and Q_C are the heat in the hot and the cold reservoirs. Hence,

$$\eta = \frac{Q_H - Q_C}{Q_H}, \tag{17.11}$$

or;

$$\boxed{\eta = 1 - \frac{Q_C}{Q_H}}. \tag{17.12}$$

We also can write;

$$Q_H = T_H \Delta S; Q_C = T_C \Delta S, \tag{17.13}$$

or;

$$\boxed{\eta = 1 - \frac{T_C}{T_H}}. \tag{17.14}$$

In equation 17.14 T_C and T_H are the temperatures of the hot and the cold reservoirs in degree Kelvin.

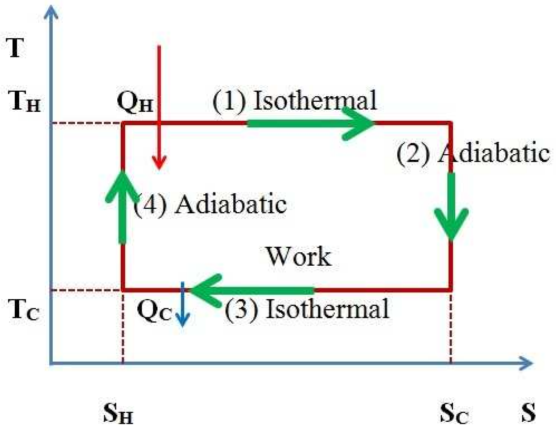

Figure 17.1: Temperature vs. entropy for an ideal engine called the *Carnot Engine* (see text above).

Note, the Carnot cycle is not obtainable because there are no fully *reversible* processes and the *entropy* always increases. The best way to illustrate a *Carnot Engine* is with the aid of a $T-S$ diagram. As we described above, we expect the *Carnot Engine* to be a cyclic machine that has no net entropy increase. Figure 17.1 shows this process.

17.4 Problems

1. Calculate the amount of the work done by a Carnot engine if the ratio of the temperature of the hot reservoir to that of the cold reservoir is 1.3 and the initial heat supplied is 10000 J.

2. Find the efficiency of an engine operating between temperatures of 250 $°C$ and 180 $°K$.

3. Find the entropy change of a reversible engine from state a to b if dQ amount of heat is transferred to the system at a temperature T.

4. A steam engine takes steam from a boiler at 250 $°C$ and releases it into the air at 75 $°C$. Calculate the efficiency of the steam engine.

5. Calculate the heat required to increase the temperature of 1.0 l of water from freezing (0 $°C$) to boiling at 100 $°C$.

6. In problem 5, assume the entropy of water at 0 $°C$ is zero, find the entropy at 100 $°C$.

7. Calculate the final temperature of 1 l of water at 0 $°C$, when mixed in with 1 l of water at 100 $°C$. What is the entropy of the final mixture?

8. Find the work done by 1.0 mol of an ideal gas expanding by a factor of two in a container due to a temperature rise of 50 $°C$. Assume the pressure remains at one atmosphere throughout the process.

9. Calculate the rise in temperature in the process described in problem 8. Find the heat transferred to the gas.

10. A heat engine operates with 5 l of an ideal gas initially at a constant atmospheric temperature and pressure (0 $°C$ and 1 atmosphere). The engine is then turned on where the gas receives 10000 J of heat from a combustion chamber and expands to 4 times its volume at constant pressure. Find the work done by the engine and the efficiency of the engine.

11. In problem 10, what happens to the rest of the heat? What is the final temperature of the gas?

12. A $1 - kW$ Carnot engine is operating between 100 $°C$ and 10 $°C$. Find the rate of heat intake and exhaust by the engine. Find the efficiency of the engine.

13. An ice cube with a mass of 20 g has a temperature of 0 $°C$ and is dropped into 150 g of water at 20 $°C$. What is the change in entropy when the system reaches equilibrium?

14. Calculate the change in entropy for one mol of neon gas which is monatomic, undergoing a constant volume process from 300 $°C$ to 350 $°C$.

15. Repeat problem 14 when the pressure is constant.

16. Calculate the efficiency of a theoretical engine operating between the Earth core temperature of 6000 $°C$ and the equator at 35 $°C$.

17. A heat engine, with efficiency of 25%, generates 500 J of work in each cycle. How much thermal energy is a) absorbed and b) released in each cycle?

18. A power plant uses sea water as its cold reservoir. It uses 400 $°C$ steam as its input heat source. If the efficiency of the plant is 20% when the sea water is 20 $°C$, calculate its efficiency when the sea water is only 10 $°C$.

19. A 1 GW nuclear power plant uses sea water as its cold reservoir. How much energy does the power plant produce per day?

20. The reactor in problem 19 has an efficiency of 20%. If the temperature of the heat reservoir is 400 $°C$, what is the exhaust temperature? What is the increase in entropy?

21. A new engine invention is claimed to produce 50 kJ of energy by using 100 kJ of heat from a reservoir and releasing 20 kJ. Would you give a patent to this invention?

Appendix

Answers to Even-Numbered Problems

Chapter 1.
1.2: $\approx 5.1 \times 10^8\ km^2$, $Radius\ of\ the\ Moon \approx 1738\ km$
1.4: $\approx 4.37\ l.y.$; $\approx 4.137 \times 10^{13}\ km$
1.6: $\$0.5/m^2$
1.8: $\approx 2.25 \times 10^{22}\ m^2$
1.10: $\rho = 19.32\ g/cm^3$
1.12: $\rho_{Earth} \approx 5.5\ g/cm^3$
1.14: $\rho_{Moon} \approx 3.35\ g/cm^3$
1.16: $\rho_{proton} \approx 3.99 \times 10^{14}\ g/cm^3$
1.18: $\rho_{H-atom} \approx 0.05\ g/cm^3$; $\frac{\rho_p}{\rho_H} = 8 \times 10^{15}$
1.20: $0.125\ g$
1.22: $\approx 4.42 \times 10^{56}\ protons$

Chapter 2.
2.2: $\approx 2.24\ km$; $\theta \approx 26.6°$
2.4: $|\mathbf{R}| \approx 24.2\ m$; $\theta \approx 12.2°$, the angle with the longer displacement vector.
2.6: $Component\ along\ the\ incline = 50.0\ N$; $Component\ normal\ the\ incline = 86.6\ N$
2.8: $|\mathbf{F}| \approx 744\ N$; $\theta \approx 82.9°$, angle with the East.

2.10: $\mathbf{R} = 4\mathbf{i} + 14\mathbf{j} - 8\mathbf{k}$; $\mathbf{D} = 2\mathbf{i} + 4\mathbf{k}$
2.12: $|\mathbf{A}| \approx 7.87$, $\theta \approx 68.8°$ and $\phi \approx -33.7°$; $|\mathbf{B}| \approx 9.27$, $\theta \approx 81.9°$ and $\phi \approx -80.5°$
2.14: -55
2.16: $20\ N.m$
2.18: $\mathbf{R} = -28\mathbf{i} + 16\mathbf{j} + 14\mathbf{k}$

Chapter 3.
3.2: $v_{average} \approx 9.26\ km/h$
3.4: $\approx 8.67\ years$; $\approx 16.7\ days$
3.6: $t \approx 27.5\ s$
3.8: $-2.25\ m/s^2$
3.14: $h = 39.6\ m$, $v \approx 29.6\ m/s$
3.16: $v \approx 14\ m/s$
3.18: $t \approx 1.2\ s$
3.20: $a = 10.0\ m/s$; $x = 11.25\ m$

Chapter 4.
4.2: $\theta \approx 82°$
4.4: $v_0 \approx 43\ m/s$
4.6: $y \approx -.333\ \mu m$
4.8: $\theta \approx 82°$; $t \approx 2.2\ s$
4.10: $\omega_{Earth} \approx 0.99\ Rad/s$; $\omega_{Mars} \approx 0.60\ Rad/s$; $\omega_{Moon} \approx 0.40\ Rad/s$
4.12: $v_{Mars} \approx 24.0\ km/s$
4.14: $g_{650}\ km) \approx 7.73\ m/s^2$
4.16: $v \approx 188.5\ m/s$
4.18: $a_{Sun}/a_{star} = 2.5$
4.20: $R \approx 1.58\ m$

Chapter 5.
5.2: $k = 500.0\ N/m$
5.4: $v = \sqrt{2gl\sin\theta}$
5.6: $v = \sqrt{\frac{2x(mg+kx)}{m}}$
5.8: $a = 2.96\ dyn$
5.10: $x \approx 14.0\ m$; $x = -2.5\ m$
5.12: $T_{12} = 20000\ N$; $T_{23} = 10000\ N$
5.14: $\omega_{max} = \infty$
5.16: $\mu_s = \frac{a}{g}$
5.18: $v_{max} \approx 13.9\ m/s$
2.20: $g \approx 24.87\ m/s^2$

Chapter 6.
6.2: $W = 49000\ N$
6.4: $F \approx 28.28\ N$
6.6: $W \approx 43.3\ kJ$
6.8: $W \approx 3.393\ MJ$; $P \approx 3.27\ kW$
6.10: $T \approx 5.83\ kJ$; $v \approx 103.9\ km/hr$
6.12: $x \approx 2.91\ cm$
6.14: $\theta \approx 48.2°$
6.16: $a = \frac{M_2}{M_1+M_2}$; No, because forces have to be balanced using Newton's laws.
6.18: $a = \frac{2gh-4gR}{R}$; $a = \frac{2gh-2gR}{R}$

Chapter 7.
7.2: $X_{c.m.} \approx 3.44\ cm$; $Y_{c.m.} \approx 2.61\ cm$
7.4: $a = \frac{p}{mt}$; $F = \frac{p}{t}$
7.6: $J = 10.0\ kg.m/s$

7.8: $F = 200.0\ N$
7.10: $V \approx 0.56\ m/s$; $T_i \approx 3136\ J,\ T_i \approx 6173\ J$
7.12: $V = \frac{m\sqrt{2gh}\sin 2\theta}{2M}$
7.14: $p_\alpha = p_{Th} \approx= 193.8\ MeV/c$; $T_\alpha = 5.04\ MeV,\ T_{Th} = 0.09\ MeV$
7.18: $x \approx 52.3\ cm$
7.20: $v_{proto-Earth} \approx 29.2\ km/s$; $d_{proto-Earth} \approx 1.515 \times 10^8\ km$

Chapter 8.
8.2: $v_{Moon} \approx 1\ km/s$; $a_c \approx 2.63 \times 10^{-3}\ m/s^2$
8.4: $a_c \approx 5.93 \times 10^{-3}\ m/s^2$
8.6: $\alpha = 18.33\ rad/s^2$
8.8: $I_{sphere} = \frac{2}{5}MR^2$
8.10: $I = \frac{1}{2}MR^2$
8.12: $a = \frac{2M}{m+2M} g\sin\theta$; $T = \frac{mM}{m+2M} g\sin\theta$
8.14: $F = Mg\frac{\sqrt{2Rh-h^2}}{R-h}$; $h = R$
8.16: $v_{hoop} = \sqrt{gh}$; $v_{sphere} = \sqrt{\frac{10}{7}gh}$
8.18: *a)* $L_{Moon} \approx 2.8 \times 10^{34}\ kg.m^2/s$; *b)* $L_{Moon} \approx 1.1 \times 10^{37}\ kg.m^2/s$; *c)* $L_{Earth} \approx 7.03 \times 10^{33}\ kg.m^2/s$
8.20: $\alpha = \frac{5\ rad/s}{I}$; $\tau = 6.25\ N.m$
8.22: $L_f = L_1 + L_2$; $T_i = \frac{1}{2}I_1{\omega_1}^2 + \frac{1}{2}I_2{\omega_2}^2$; $T_f = \frac{1}{2}(I_1 + I_2){\omega_f}^2$

Chapter 9.
9.2: $R_1 = 2744\,N,\ R_2 = 2940\ N$
9.4: $T_{BC} = 60.0\ N$
9.6: Overhang for *i)* second brick = l/6; *ii)* third brick = l/4; *iii)* fourth brick = l/2
9.8: $\mu_s = \frac{1}{2}\cos\theta$
9.10: $\sigma_{AC} \approx 90.1\,MPa$; $\epsilon_{AC} \approx 4.5\times10^{-4}$; $\sigma_{BC} \approx 156.0\ MPa$; $\epsilon_{BC} \approx 7.8\times10^{-4}$
9.12: $d = 0.11\ mm$
9.14: $\beta \approx 30.49°$
9.16: $W \approx 136660\ N$

Chapter 10.
10.2: $d = 258397\ km$
10.4: $t \approx 1.914\ h$; $v \approx 11\ km/s$
10.6; $F \approx 10^{-47}\ N$
10.8: $v \approx 24.2\ km/s$
10.10: $g_{d=-1000\ km} \approx 8.33\ m/s^2$
10.12: $M. \approx 2.0 \times 10^{30}\ kg$
10.14: Problem 13 gives more accurate results. For more accurate results, we must include the masses of the Moon and Phobos.
10.16: $h \approx 592\ km$
10.18: $g = \frac{2GM(1-z/\sqrt{(R^2+z^2)}}{r^2}$
10.20: $U \approx -1.6 \times 10^{48}\ J$

Chapter 11
11.2: $f = 2.0\ Hz$; $\omega \approx 12.57\ Rad/s$
11.4: $v = -5.0\pi\cos(\pi t\ m/s + \pi/2)$; $a = -5.0\pi^2\sin(\pi t + \pi/2)\ m/s^2$; $x = 0,\ v = 5.0\pi\ m/s, a = 0$
11.6: $t \approx 74\ ms$
11.10: $g \approx 9.675\ m/s^2$
11.12: $l_{Mars} \approx 37\ cm$; $l_{Moon} \approx 17\ cm$
11.14: $\tau_{pole} = 2.0\ s; \tau_{equator} \approx 2.01\ s$

Chapter 12
12.2: $F = 14.7\ N$
12.4: $F = 0.288\ N$
12.6: $v = 3.2\ m/s$
12.8: $Distance\ from\ the\ edge \approx 1.37\ m$
12.10: $d \approx 6.15\ g/cm^3$
12.12: $\rho = \frac{l}{l+h}$
12.14: $p = 98.0\ Pa$
12.18: $Thrust \approx 157080\ N$

Chapter 13
13.2: $\nu = 5 \times 10^5\ Hz;\ \omega = 10^6 \pi\ rad/s$
13.4: $v \approx 77.8\ m/s$
13.6: $v \approx 4.47\ m/s$
13.8: $\mu = 0.005\ kg/m;\ T = 264.5\ N$
13.10: $I_{d=10\ m} \approx 0.318\ W/m^2$ 13.12: $I \approx 3.2 \times 10^{10}\ \nu/cm^2$
13.14: $v \approx 8.1\ m/s;\ \nu \approx 3.12\ Hz$

Chapter 14
14.2: $d = 1700\ m$
14.4: $d \approx 4472\ m$
14.6: $v = 1000\ m/s$
14.8: $v \approx 4458\ m/s$
14.10: Distance to the first two buoys $\approx 741\ m$, and distance to the third buoy $\approx 1037\ m$
14.12: $\nu = 550\ Hz$
14.14: $\nu \approx 8133\ Hz$
14:16: $v \approx 44.72\ m/s$

Chapter 15
15.2: $t = 1000\ ^\circ F$
15.4: $\Delta L = 27\ cm$; so the gap is not wide enough.
15.8: $d \approx 2.999\ cm$
15.10: $F_{minimum} \approx 65000\ N$
15.14: $\Delta t < 721\ ^\circ C$
15.16: $T = e^{-Kt}$ 15.18: $t \approx 0.15\ ^\circ C$
15.20: $t \approx 0.54\ ^\circ C$
15.22: The light has to be on for 42.6 hours!
15.24: $Q \approx 3471\ kJ$
15.26: $Q \approx 502\ kJ$
15.28: $\Delta U = 5.0\ kJ$
15.30: $m \approx 1.66\ kg$

Chapter 16
16.2: $Q \approx 2270\ J$
16.4: $p \approx 3101\ Pa$
16.6: $T \approx 154.6\ ^\circ K$
16.8: $\Delta v \approx 50\ m/s$
16.10: $W = 10.1325\ kJ$
16.12: $U \approx 174\ J$
16.14: $N \approx 7.5 \times 10^{13}\ mol$
16.16: $N \approx 246\ atoms/cm^3$

Chapter 17
17.2: $\eta \approx 66\%$
17.4: $\eta \approx 33\%$
17.6: $\Delta S = 4182\ J/^\circ K$
17.8: $W \approx 415.7\ J$
17.10: $W \approx 1520\ J$; $\eta \approx 15.2\%$
17.12: $rate \approx 760\ W;\ 240\ W$; $\eta \approx 24\%$
17.14: $\Delta S = 8.314\ J/^\circ K$
17.16: $\eta \approx 95\%$
17.18: $\eta \approx 21.4\%$
17.20: $T_C = 265.4\ ^\circ C$

About the author

The Author, Ali R. Fazely, attended Oklahoma State University, where he received the degree of Bachelor of Science in Mechanical Engineering in 1975. In 1977, he received a Master of Science degree in Mechanical Engineering also from Oklahoma State University. In 1980, he received a Master of Arts in Physics from Kent State University. He received his Ph.D. in Experimental Intermediate Nuclear Physics from Kent State University in 1982. After a brief stay as a postdoctoral researcher at Kent, he accepted a postdoctoral position at the Louisiana State University in February 1983. He participated in several Nuclear and Particle Physics experiments at the Los Alamos National Laboratory as a participant and as a spokesperson, where he was a visiting scientist from LSU between 1984 and 1990. In 1990, he moved back to LSU as a Research Assistant Professor and in 1991, he accepted an Associate Professor position at Southern University where he established the High Energy and Astrophysics Group. At the present time, he is a retired professor of Physics at Southern and is involved in neutrino astronomy with the IceCube detector at the South Pole. His interest is to study the structure of the universe using neutrinos as cosmic messengers and various properties of the neutrino and overlap of science and religion.